Spectral properties of non-selfadjoint rational operator functions and applications to wave propagation in photonic crystals

Axel Torshage & Christian Engström
Department of Mathematics and Mathematical Statistics

Problem statement

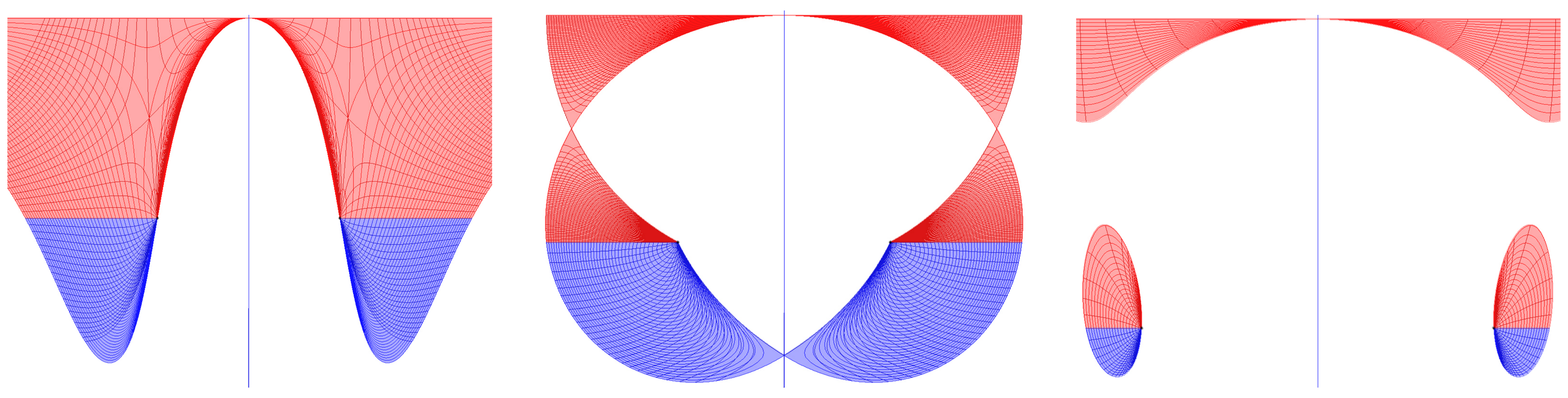
The operator valued function $\mathcal{T}:\mathcal{H}\to\mathcal{H}$ is for Hilbert space $\mathcal{H}$ defined as

$$\mathcal{T}(\omega) := A-\omega^2 - B\frac{c-\mathrm{i}d\omega}{c-\mathrm{i}d\omega-\omega^2}B^*, \quad \operatorname{dom}\mathcal{T} = \operatorname{dom}A,$$

$$\text{where} \quad \omega\in\mathbb{C}\setminus\{\delta,-\bar{\delta}\}, \quad \theta := \sqrt{c-\frac{d^2}{4}}, \quad \delta := \theta - \mathrm{i}\frac{d}{2}.$$

- A is unbounded and self-adjoint in $\mathcal{H}$.
- $B:\widehat{\mathcal{H}}\to\mathcal{H}$ is bounded.
- For each ω the operator $\mathcal{T}(\omega)$ is a linear operator in $\mathcal{H}$.
- The constants satisfies $2\sqrt{c} > d \geq 0$.

Due to non-selfadjointness of the problem the spectrum of $\mathcal{T}$ is not real and no variational principles for the eigenvalues exist, instead analytic bounds of the numerical range of $\mathcal{T}$ will be deduced.



Some examples of the bounds.

Obtaining bounds from linearization and a Krein space

The rational spectral value problem given by $\mathcal{T}$ has for $\mathbb{C}\setminus\{\delta,-\bar{\delta}\}$ equivalent spectrum to the block operator matrix $\mathcal{A}:\mathcal{H}\oplus\mathcal{H}\oplus\widehat{\mathcal{H}}\oplus\widehat{\mathcal{H}}\to\mathcal{H}\oplus\mathcal{H}\oplus\widehat{\mathcal{H}}\oplus\widehat{\mathcal{H}}$ defined as

$$\mathcal{A} := \begin{pmatrix} & A & \frac{\delta}{\sqrt{2\theta}}B & \frac{\bar{\delta}}{\sqrt{2\theta}}B \\ I & & & \\ & \frac{\delta}{\sqrt{2\theta}}B^* & \delta & \\ & -\frac{\bar{\delta}}{\sqrt{2\theta}}B^* & & -\bar{\delta} \end{pmatrix}, \quad \operatorname{dom}\mathcal{A} = \mathcal{H}\oplus\operatorname{dom}A\oplus\widehat{\mathcal{H}}\oplus\widehat{\mathcal{H}}.$$

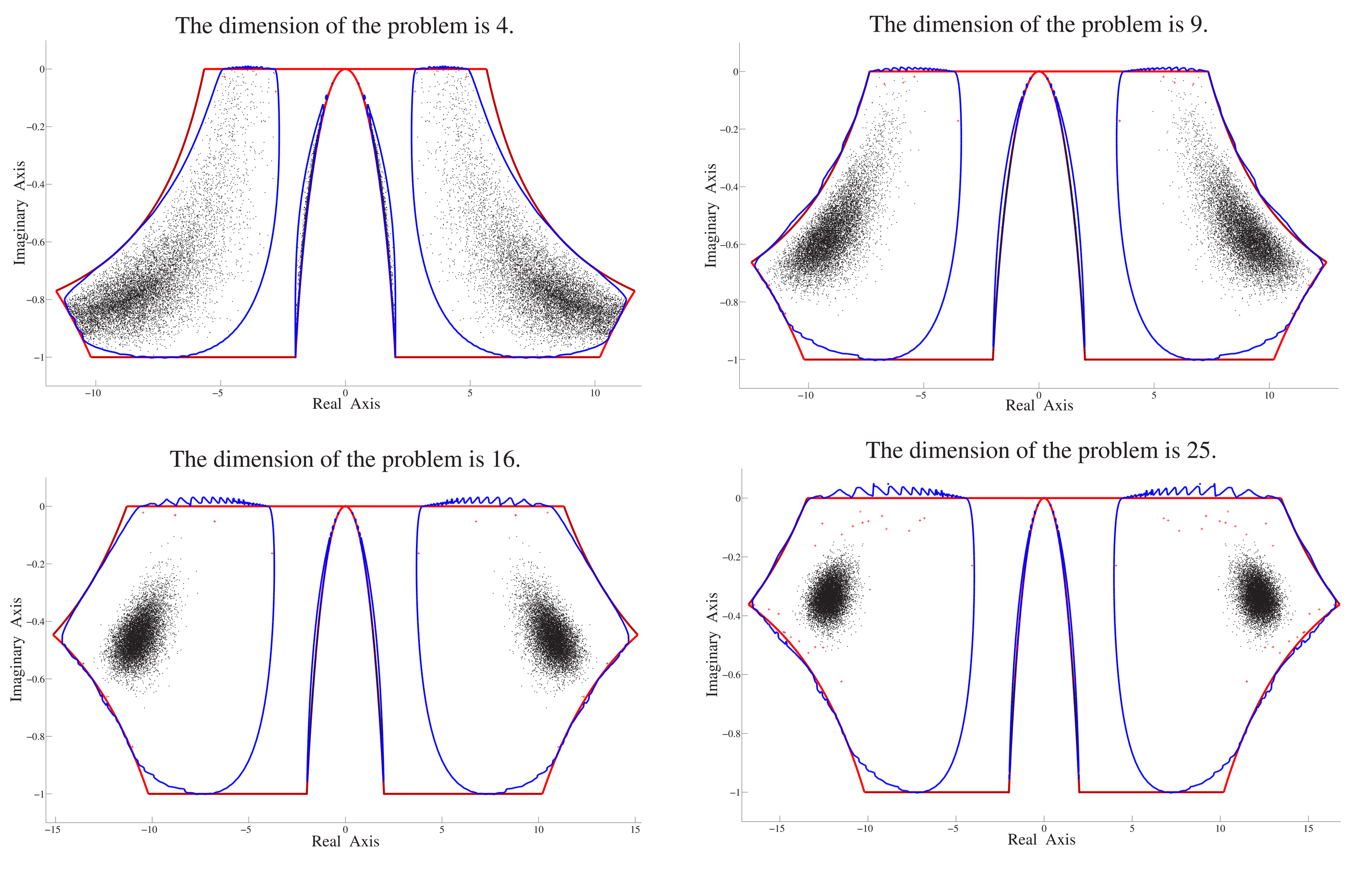
Using almost self-adjointness of $\mathcal{A}$ in a related Krein space it can be shown that the imaginary part of any eigenvalue ω to $\mathcal{A}$, as well as each point in the numerical range of $\mathcal{T}$ satisfies the implicit inequalities

$$-\frac{d}{2}\frac{\sigma_{\min}(BB^*)|\omega|^2}{|\omega|^2(|\omega+\mathrm{i}d|^2-2c)+c\left(\left(2\omega_{\Im}+\frac{d}{2}\right)^2+c-\frac{d^2}{4}\right)+\sigma_{\min}(BB^*)c} \geq$$

$$\omega_{\Im} \geq -\frac{d}{2}\frac{\sigma_{\max}(BB^*)|\omega|^2}{|\omega|^2(|\omega+\mathrm{i}d|^2-2c)+c\left(\left(2\omega_{\Im}+\frac{d}{2}\right)^2+c-\frac{d^2}{4}\right)+\sigma_{\max}(BB^*)c}.$$

Size of the bounds and the numerical range

The figures shows the bounds in red and the approximated numerical range as the blue lines and black dots for an operator valued function $\mathcal{T}$.



From the figures it follows that the numerical range is not much smaller than the presented outer bounds.

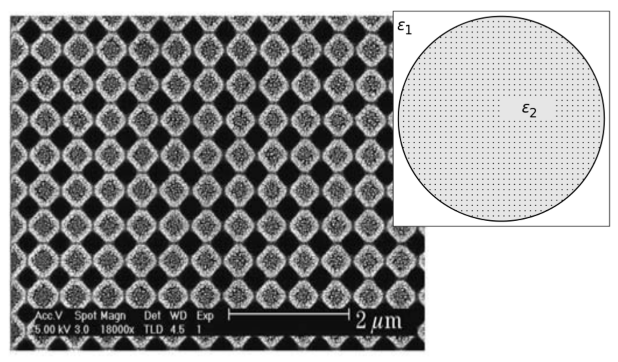
Application to photonic crystals

When studying electromagnetic wave propagation in periodic 2D-structure the vector valued Maxwell's equations can be reduced to two scalar equations. These are called the transverse magnetic polarized wave (TM) and the transverse electric polarized wave (TE).

The TM-wave can be modeled by the equation

$$\mathcal{L}E_3 := -\Delta E_3 - \omega^2\epsilon(x,\omega)E_3 = 0, \quad x := (x_1,x_2)\in\mathbb{R}^2, \quad \omega\in\mathcal{D}\subset\mathbb{C},$$

where ϵ is the permittivity function. This function is piecewise constant in space and periodic on the unit cell, from this Bloch solutions can be found by looking at only one cell instead of all of $\mathbb{R}^2$.



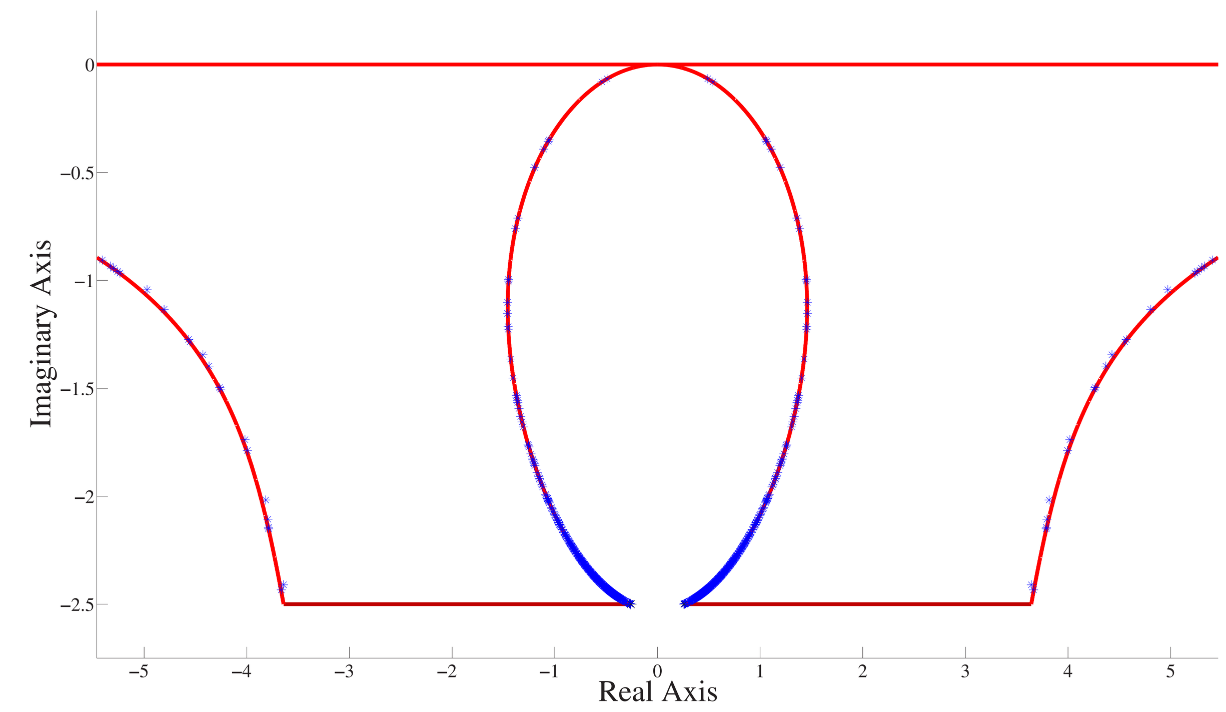
Each cell consists of two different materials, the permittivity function is constant on both materials.

The Lorentz/Drude model with frequency dispersion and absorption is studied, for whom the permittivity function is

$$\epsilon_1 = a_1, \quad \epsilon_2(\omega) = a_2 + \frac{b}{c-\mathrm{i}d\omega-\omega^2},$$

where all the constants are positive.

The accuracy of the bounds



The figure shows the bounds and the eigenvalues for a problem given by the presented application, the eigenvalues are located closely to the lower bound.

Computation time for the bounds and the numerical range

The time used to compute the bounds of $\mathcal{T}$ exactly is only dependent on finding the highest and lowest spectral value of A and BB^*, if these are known beforehand the computation is instantaneous and only plotting time has too be considered which is negligible.

This compares favorably to methods of approximating the smaller numerical range since those methods are usually only approximations and greatly depends on the dimension of the problem and are thus unable to manage infinite dimensional problems.

	Random vectors	Edge following method	The bounds
$n=4$	$7.1s(10000pts)$	$16.6s(h=0.05)$	$0.06s(1000pts)$
$n=9$	$8090s(10^8pts)^{(1)}$	$97.5s(h=0.025)$	$0.06s(1000pts)$
$n=16$	X	$193s(h=0.025)$	$0.06s(1000pts)$
$n=25$	X	$495s(h=0.025)$	$0.06s(1000pts)$
$n=36$	X	$925s(h=0.025)$ $4950s(h=0.010)$	$0.06s(1000pts)$
$n=49$	X	X	$0.07s(1000pts)$
$n=90000$	X	X	$33s(1000pts)$
$n=\infty$	X	X	$0.06^{(2)}s(1000pts)$

The table compares the time consumption for a number of different discretizations of the infinite dimensional application.

1. The approximated numerical range still was unsatisfactorily.
2. For $n=\infty$ the highest and lowest spectrum values are known and thus there is only plotting time.