A Menu of Stationary Solutions to the Axisymmetric Einstein-Vlasov System

Ellery Ames, Håkan Andreasson, Anders Logg
Chalmers University of Technology

CHALMERS

FEniCS project

Questions

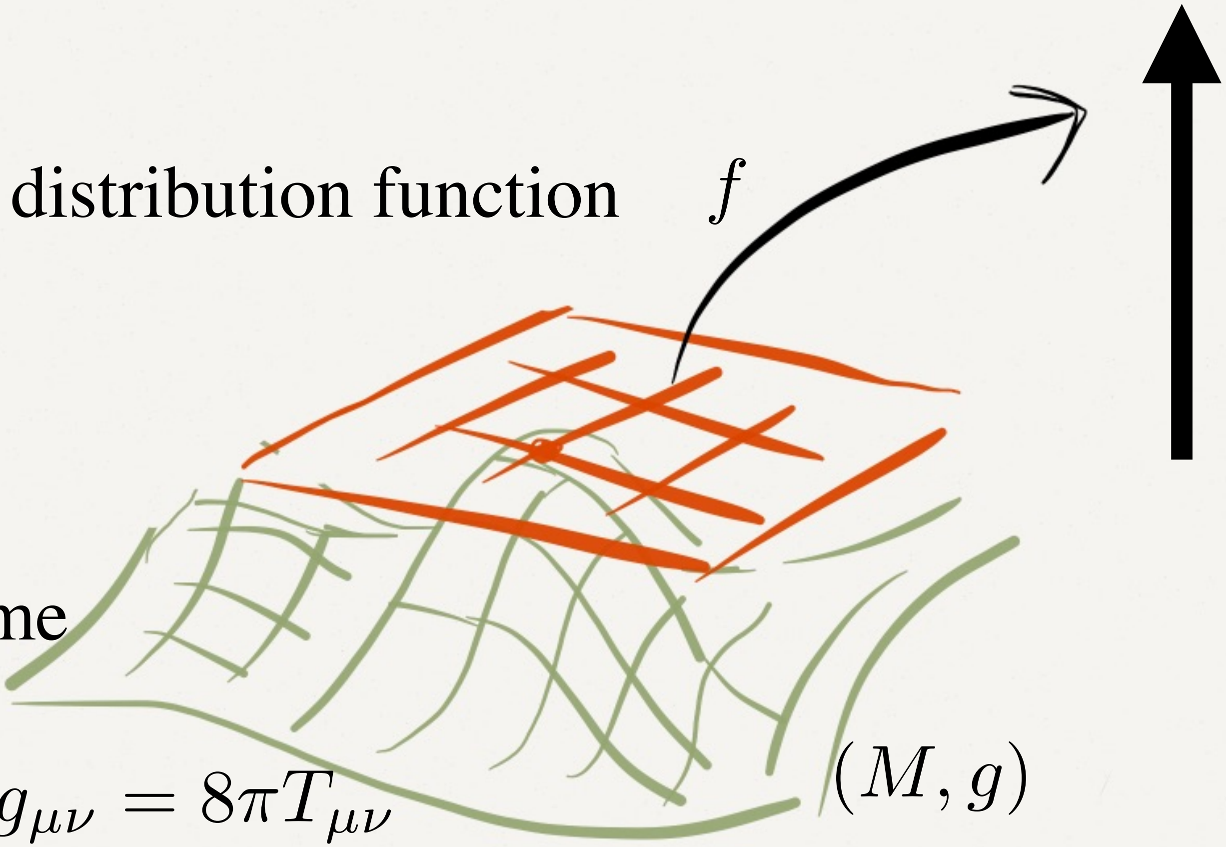
- Can we model the large variety of galaxies and star clusters observed?

- Can we find solutions which are far from spherically symmetric?

- Do general relativistic effects play an important role?

- Do the physical properties of these solutions match those of observed galaxies?

Model: Einstein-Vlasov



Dynamical spacetime

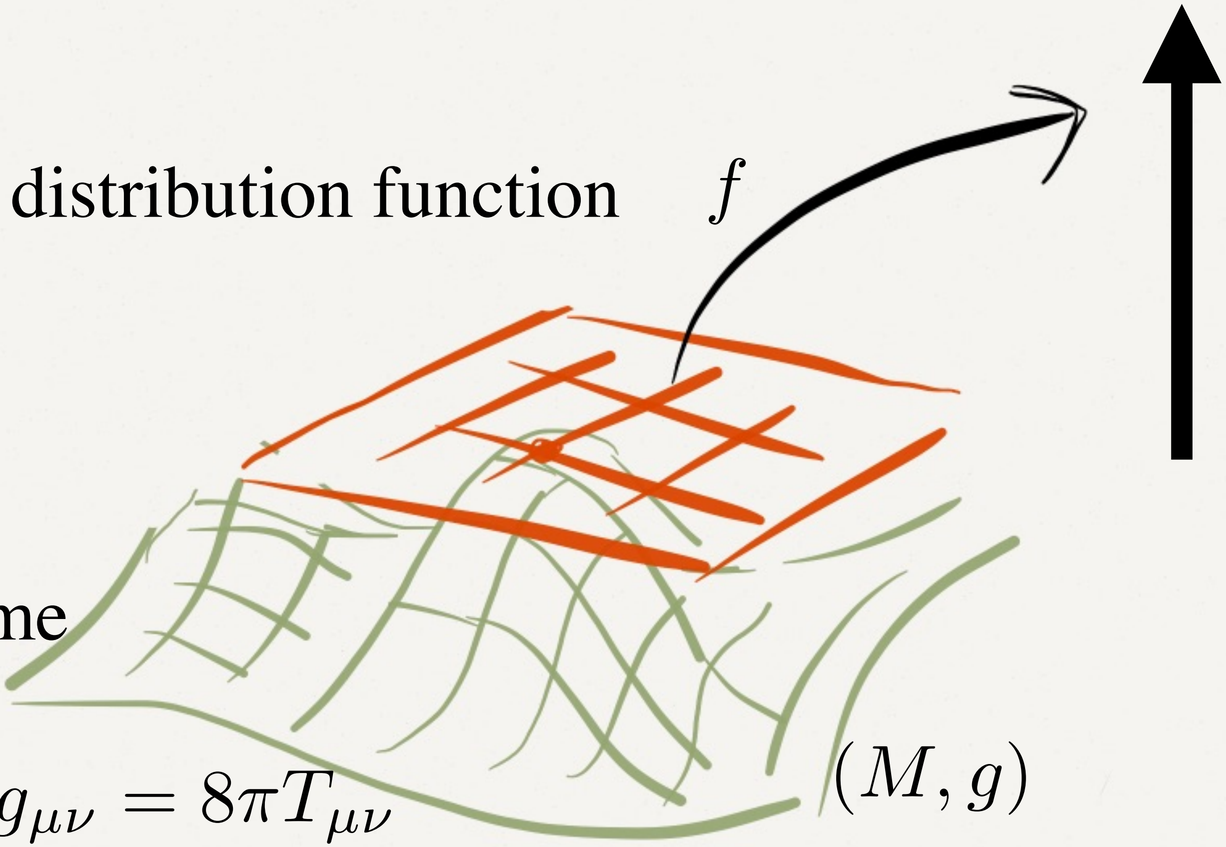
$$Ric(g)_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Vlasov equation:

$$p^\alpha \partial_{x^\alpha} f - \Gamma^a_{\mu\nu} p^\mu p^\nu \partial_{p^a} f = 0$$

Model: Einstein-Vlasov

Vlasov distribution function f



Dynamical spacetime

$$Ric(g)_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Vlasov equation:

$$p^\alpha \partial_{x^\alpha} f - \Gamma^a_{\mu\nu} p^\mu p^\nu \partial_{p^a} f = 0$$

Energy Momentum

$$T_{\mu\nu}(x) := \int_{P_x} p_\mu p_\nu f(x,p) \frac{\sqrt{-\det g(x)}}{-p_0} dp^1 \wedge dp^2 \wedge dp^3$$

Model Reductions

Assumptions

1. Equilibrium solution

2. Axisymmetry

3. All stars have the same mass

Consequences

1. Fields are independent of time

2. Fields depend only on cylindrical type coordinates (s,z)

3. Only one distribution function is needed

Ansatz

Distribution is a function of the conserved E and L

$$f(x,v) = K\Phi(E, L_z)$$

Vlasov equation is automatically satisfied

E and L_z can be written in term of the metric fields

$$E := -g(\partial_t, p^\alpha) \text{ and } L_z := g(\partial_\varphi, p^\alpha)$$

The Einstein equations become a
semilinear second order integro-differential system

Einstein-Vlasov System

$$\Delta\nu = 4\pi\left(\Phi_{00} + \Phi_{11} + \left(1 + (\rho B)^2 e^{-4\nu}\omega^2\right)\Phi_{33} + 2e^{-4\nu}\omega\Phi_{03}\right) \quad (1)$$
$$- \frac{1}{B}\nabla B \cdot \nabla\nu + \frac{1}{2}e^{-4\nu}(\rho B)^2\nabla\omega\cdot\nabla\omega$$

$$\Delta B = 8\pi B\Phi_{11} - \frac{1}{\rho}\nabla\rho\cdot\nabla B \quad (2)$$

$$\Delta\mu = -4\pi\left(\Phi_{00} + \Phi_{11} + \left((\rho B)^2 e^{-4\nu}\omega^2 - 1\right)\Phi_{33} + 2e^{-4\nu}\omega\Phi_{03}\right) \quad (3)$$
$$+ \frac{1}{B}\nabla B\cdot\nabla\nu - \nabla\nu\cdot\nabla\nu + \frac{1}{\rho}\nabla\rho\cdot\nabla\mu + \frac{1}{\rho}\nabla\rho\cdot\nabla\nu + \frac{1}{4}e^{-4\nu}(\rho B)^2\nabla\omega\cdot\nabla\omega$$

$$\Delta\omega = \frac{16\pi}{(\rho B)^2}\left(\Phi_{03} + (\rho B)^2\omega\Phi_{33}\right) - \frac{3}{B}\nabla B\cdot\nabla\omega + 4\nabla\nu\cdot\nabla\omega - \frac{2}{\rho}\nabla\rho\cdot\nabla\omega \quad (4)$$

where

$$\Phi_{00} = \frac{2\pi}{B}e^{2\mu-2\nu}\int_{e^\nu}^{\infty}\int_{-s_l}^{s_l} E^2\Phi(E,\rho s)\,ds\wedge dE,$$
$$\Phi_{11} = \frac{2\pi}{B^3}e^{2\mu+2\nu}\int_{e^\nu}^{\infty}\int_{-s_l}^{s_l}\left(s_l^2 - s^2\right)\Phi(E,\rho s)\,ds\wedge dE,$$
$$\Phi_{33} = \frac{2\pi}{B^3}e^{2\mu+2\nu}\int_{e^\nu}^{\infty}\int_{-s_l}^{s_l} s^2\Phi(E,\rho s)\,ds\wedge dE,$$
$$\Phi_{03} = -2\pi\rho B^{-1}e^{2\mu+2\nu}\int_{e^\nu}^{\infty}\int_{-s_l}^{s_l} sE\Phi(E,\rho s)\,ds\wedge dE,$$

and $s_l := Be^{-\nu}\sqrt{e^{-2\nu}(E-\omega\rho s)^2 - 1}$

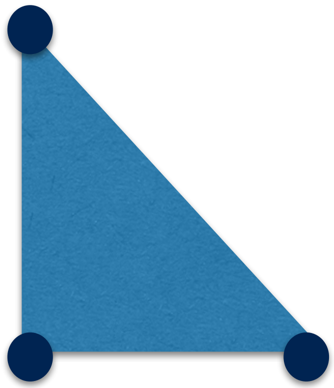
Numerical Methods: iteration

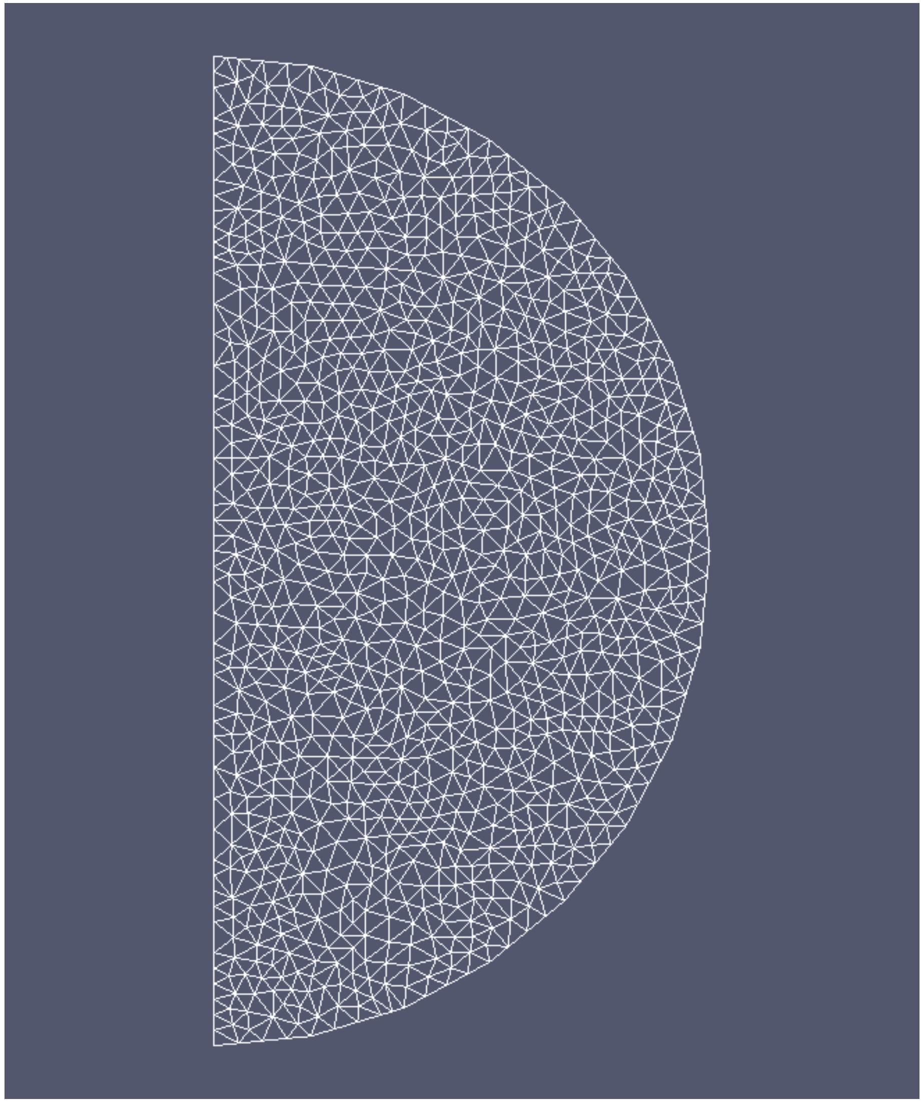
- Make ansatz of form $f = K\Phi(E, L)$
- Start with initial guess for metric fields $\left\{\nu^{(0)}, B^{(0)}, \mu^{(0)}, \omega^{(0)}\right\}$
- Compute density and mass of corresponding distribution
- Rescale ansatz to fix total mass, M
- Apply boundary conditions
- Solve linear system of PDE, obtain new fields $\left\{\nu^{(1)}, B^{(1)}, \mu^{(1)}, \omega^{(1)}\right\}$
- Compute residual, if residual < TOL break

Numerical Methods:
Finite elements

$$\Delta u + A\rho^{-1}\partial_\rho u = F(\Phi(u), u, \nabla u)$$

Piecewise-linear elements



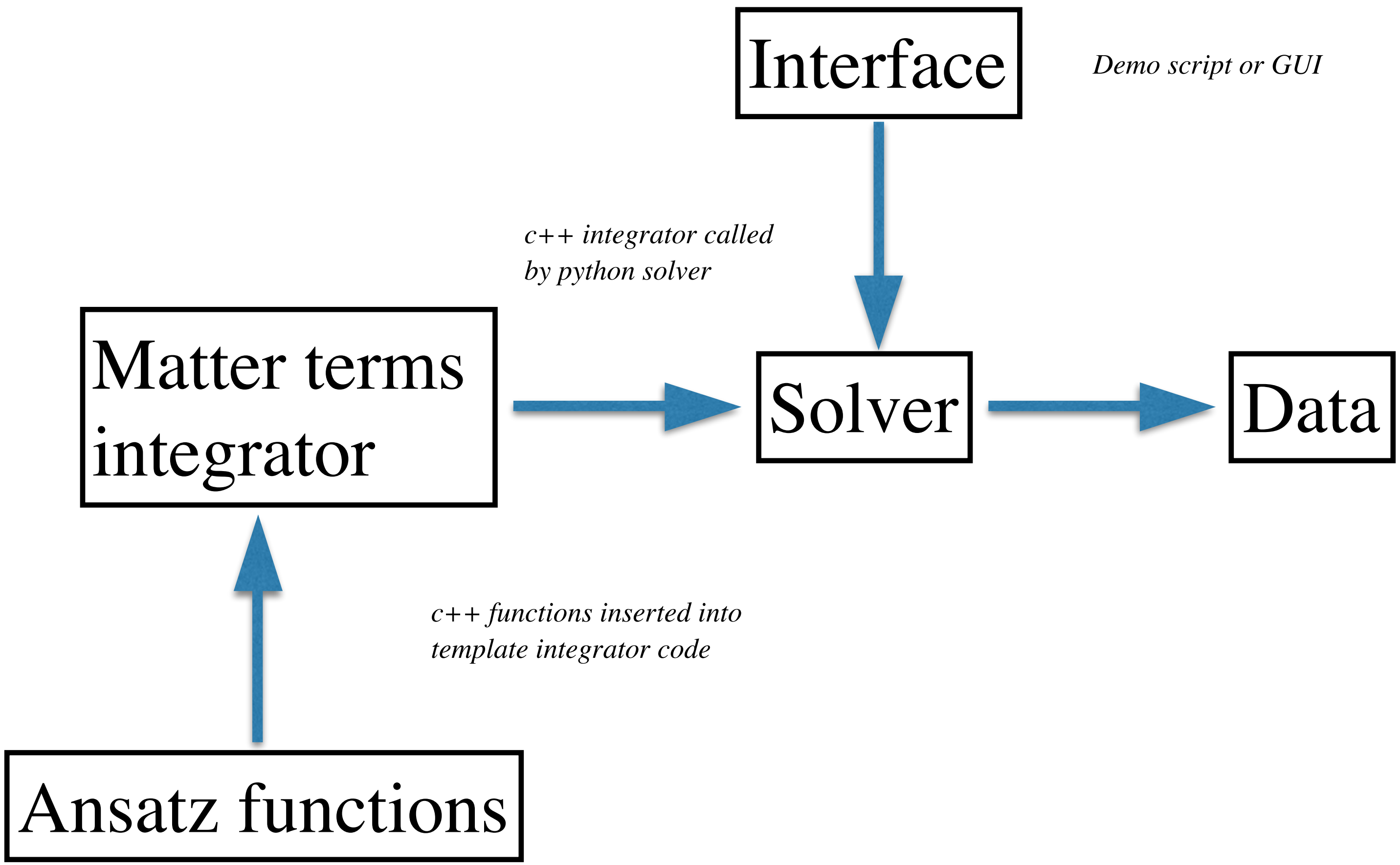


at k^th step:

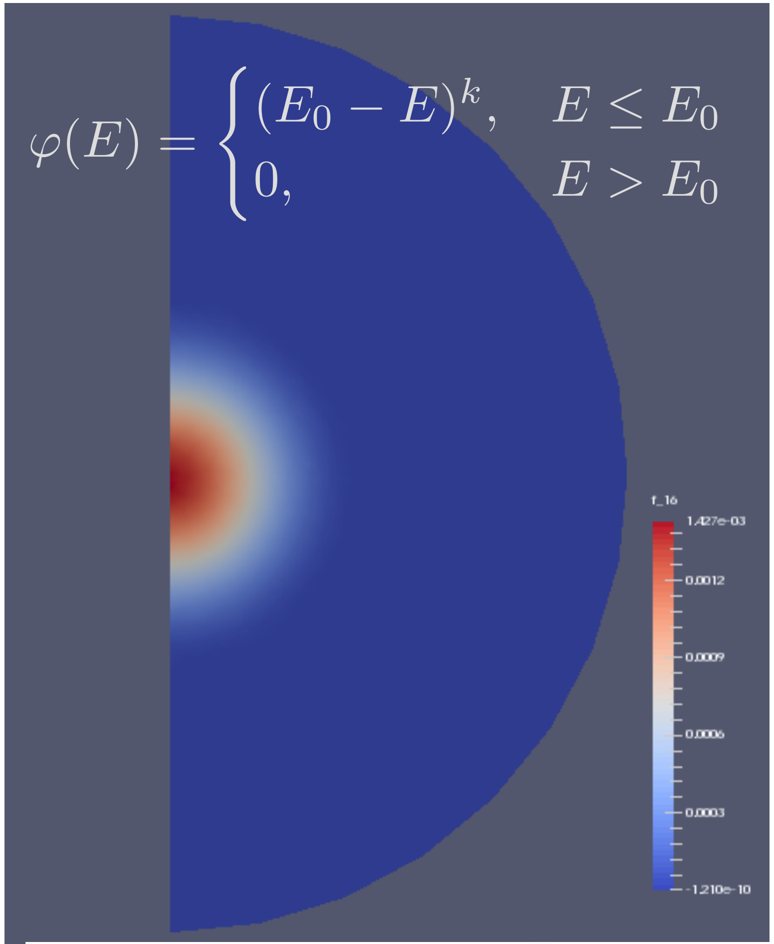
$$f^{(k)} \equiv F(\Phi(u^{(k)}), u^{(k)}, \nabla u^{(k)})$$

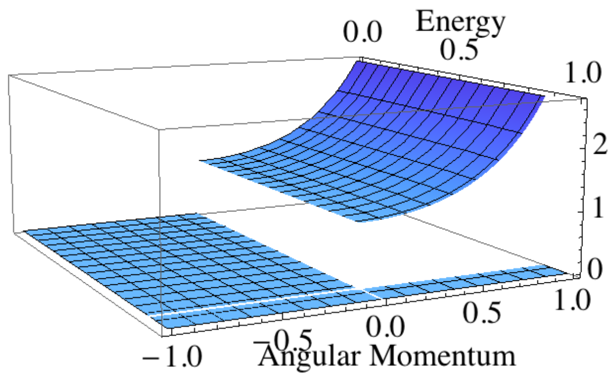
$$\left\langle \nabla u_h^{(k+1)}, \rho \nabla v \right\rangle_{L^2} + \left\langle A \partial_\rho u_h^{(k+1)}, v \right\rangle_{L^2} = \left\langle f^{(k)}, \rho v \right\rangle_{L^2},$$

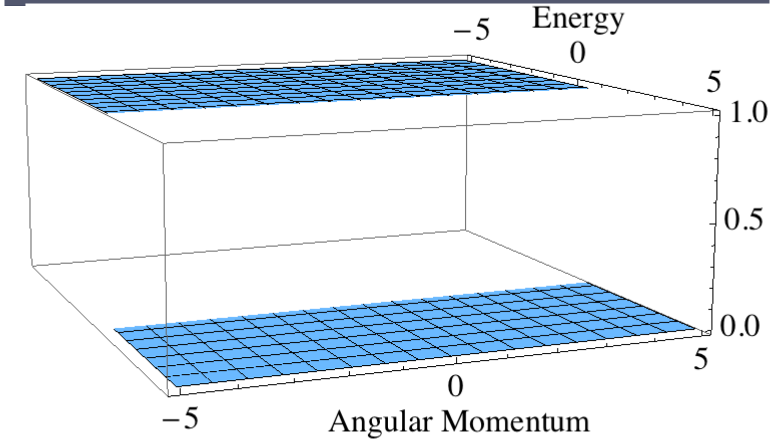
Numerical Methods: overview

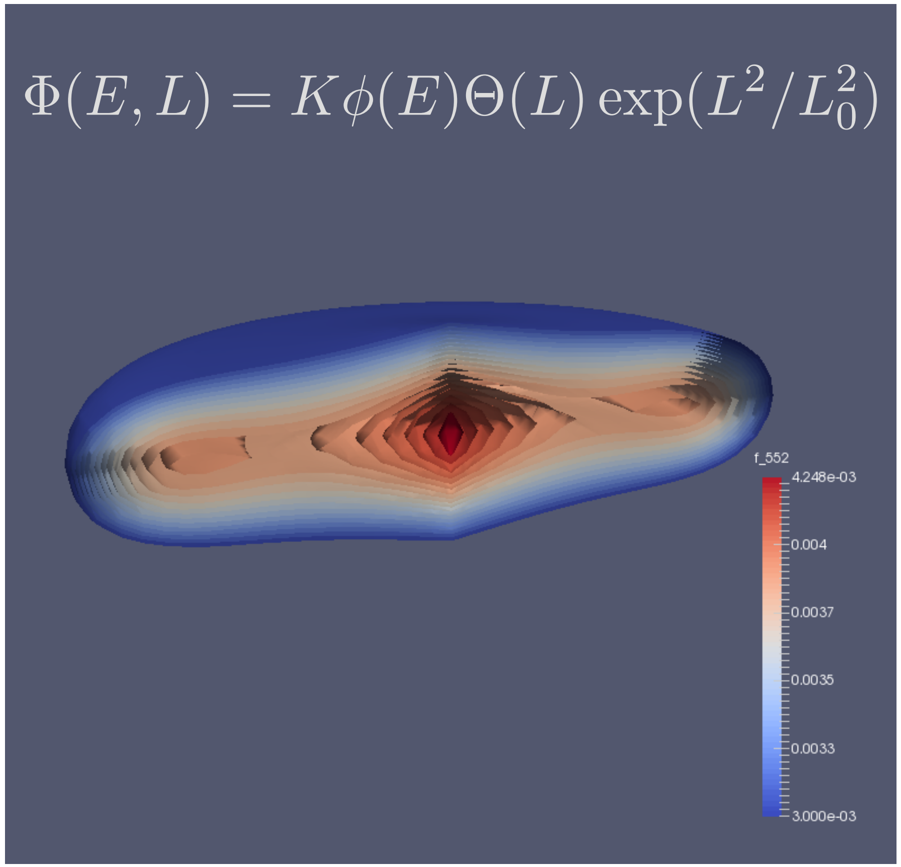


Numerical Menu of Solutions

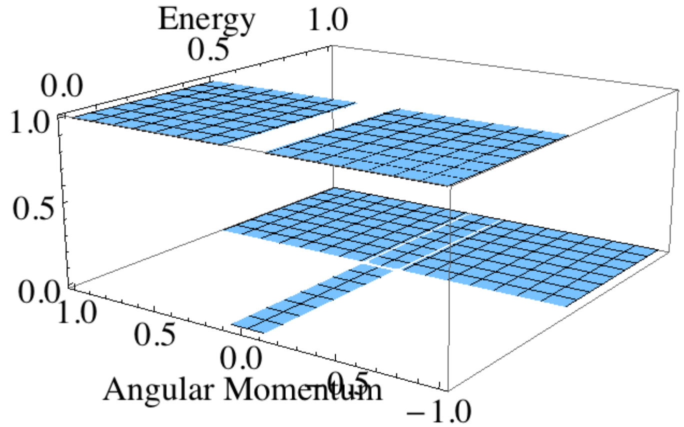


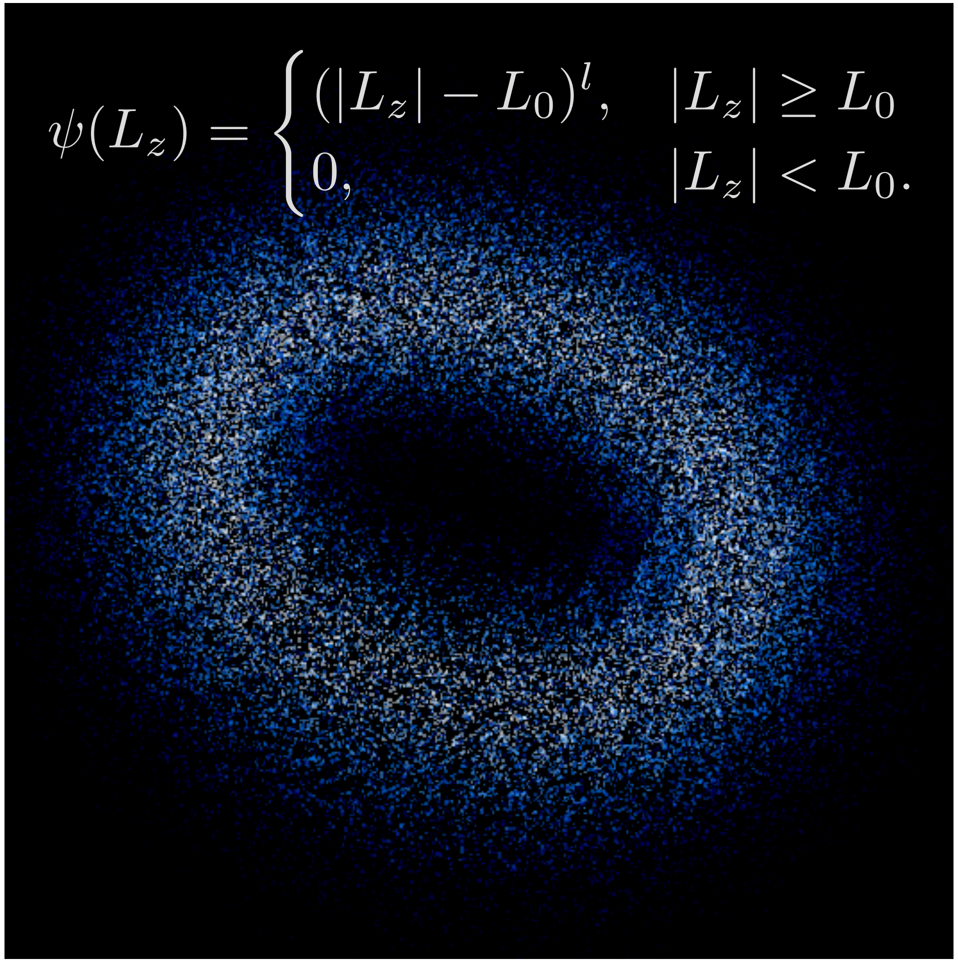


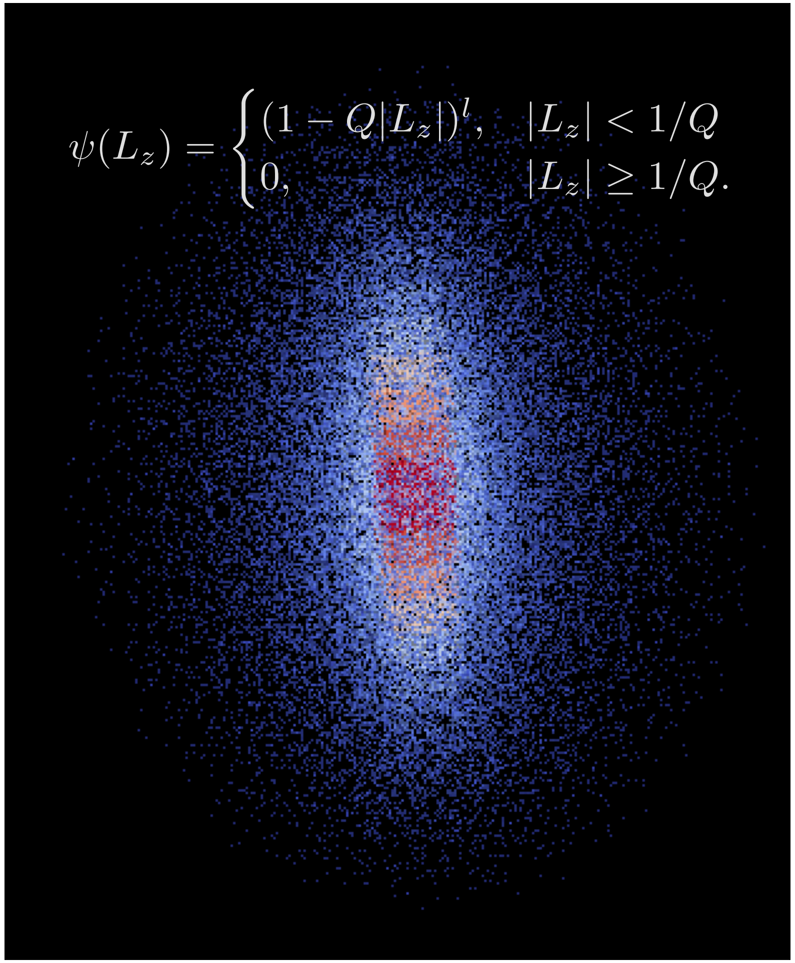


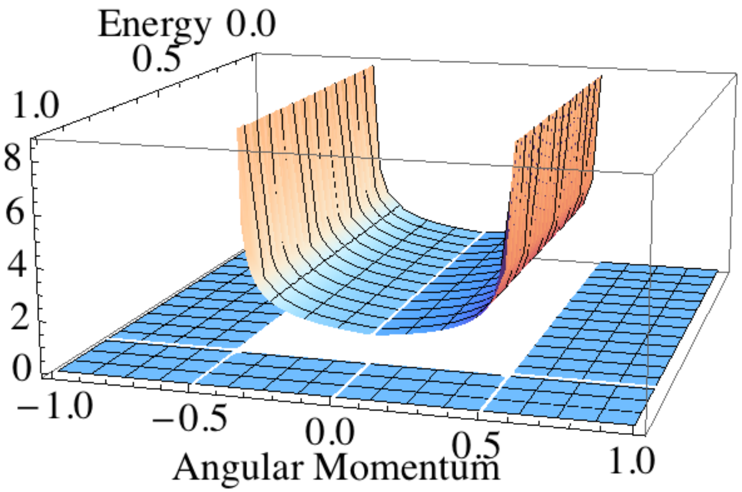


Numerical Menu of Solutions



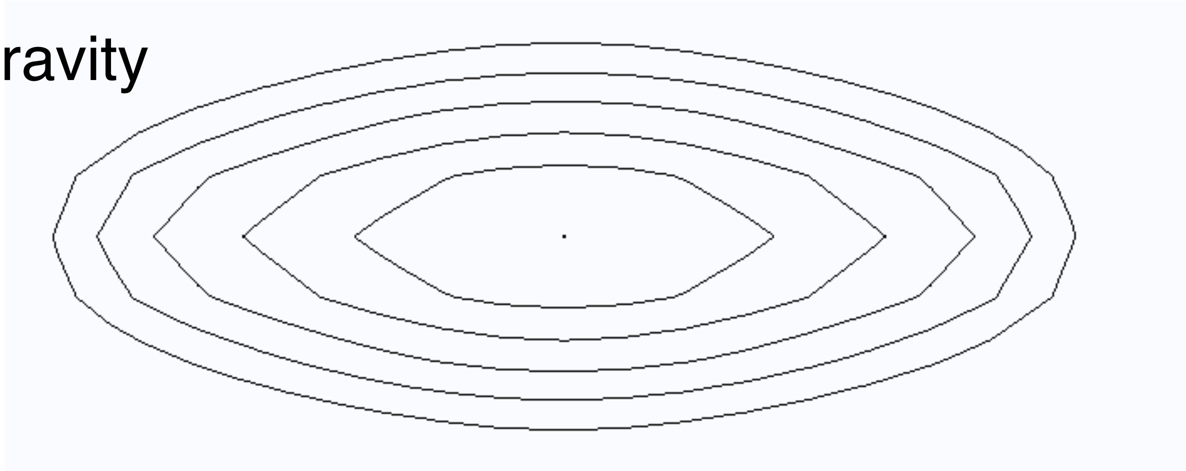


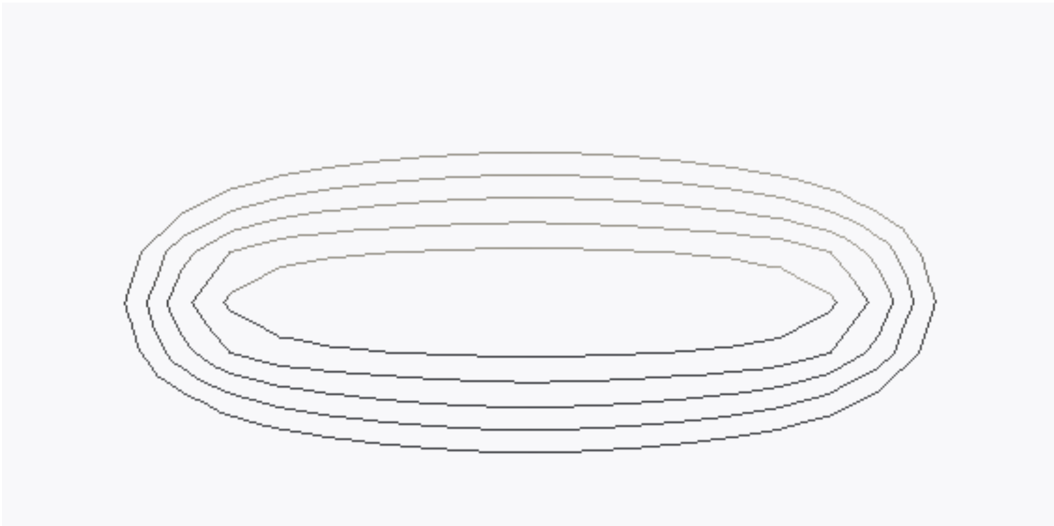




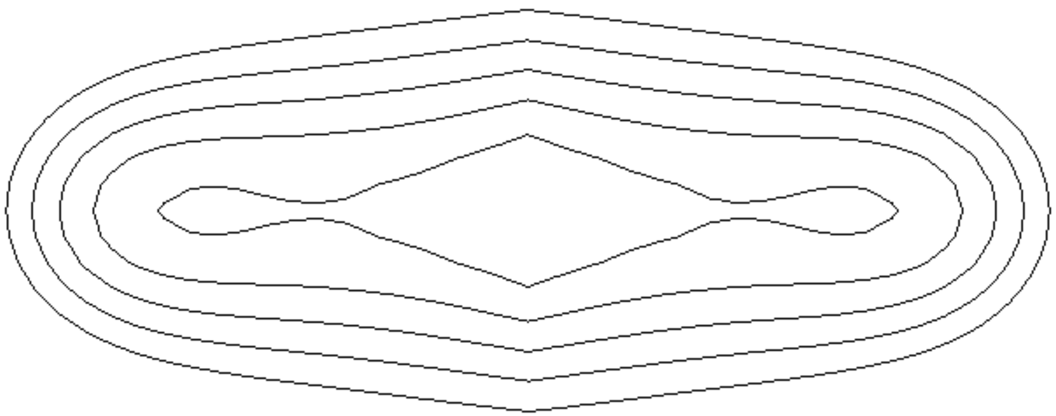
Does Relativity Matter?

Newtonian gravity



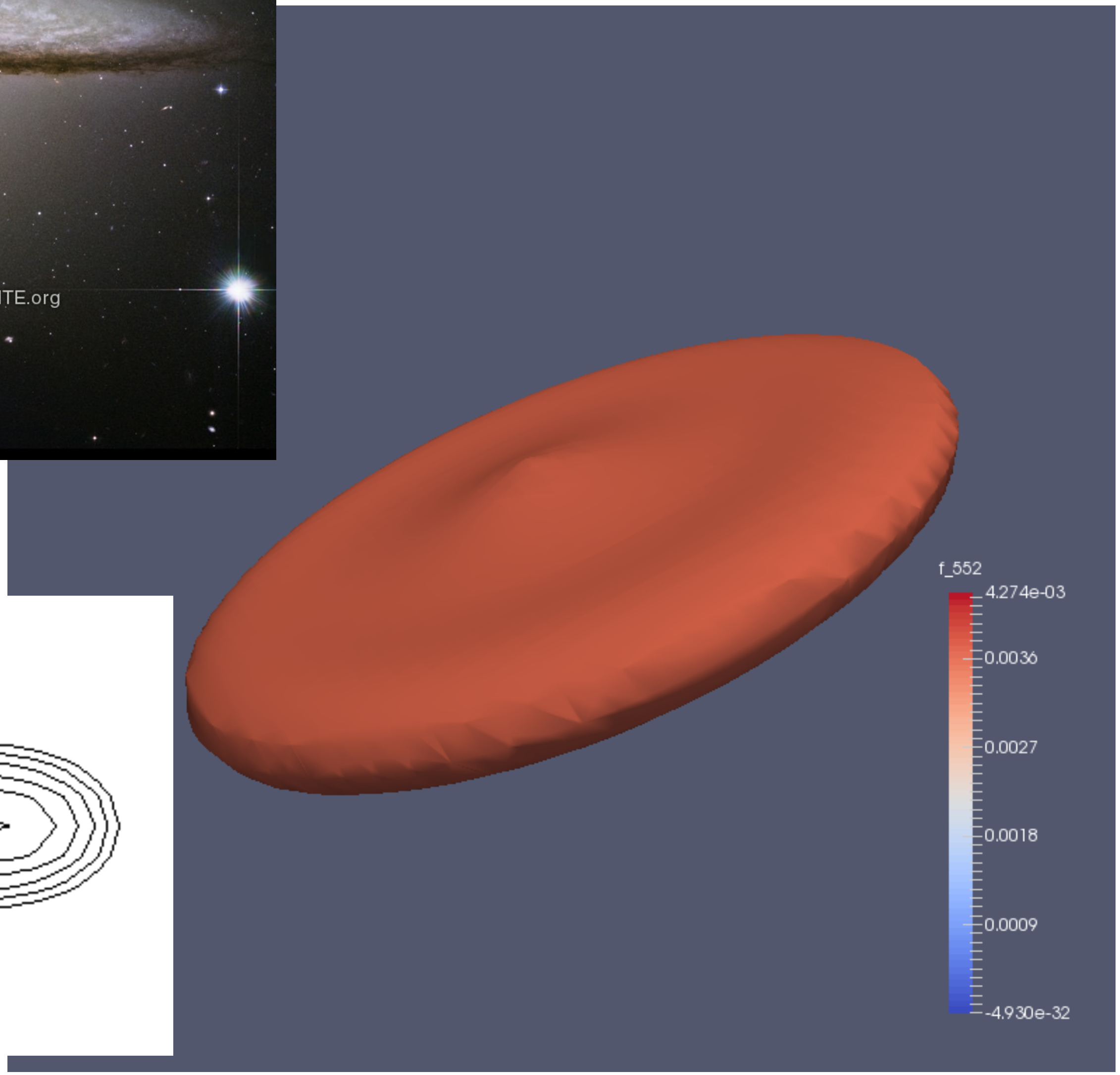


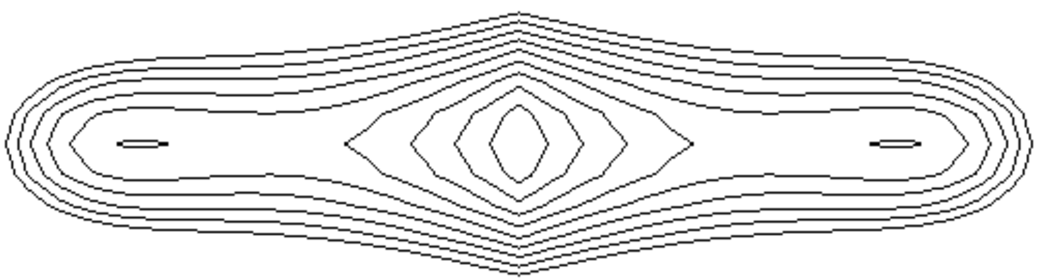
Non rotating relativistic disk galaxy



Rotating relativistic disk galaxy

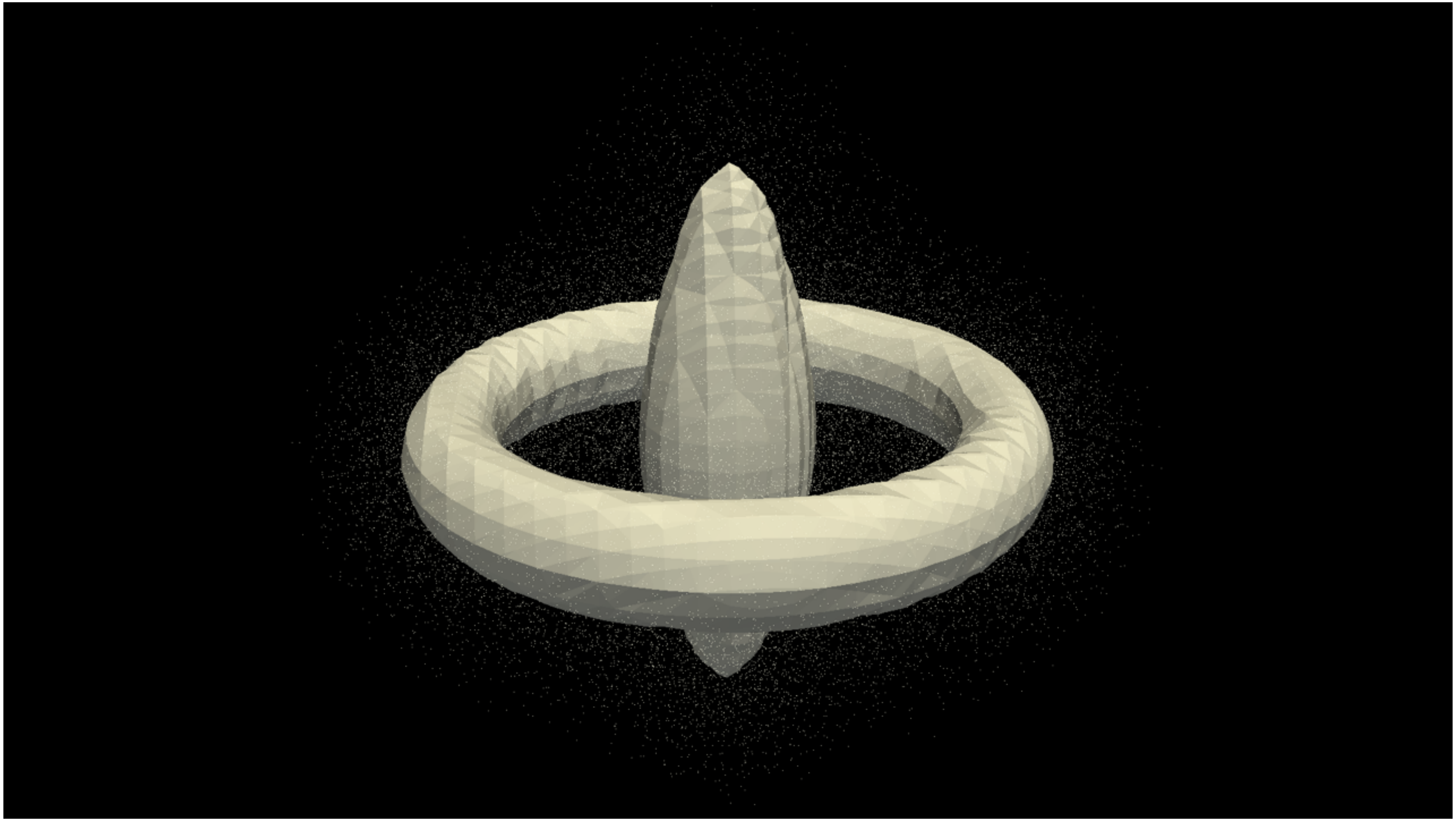

The Sombrero Galaxy — NGC 4594 (M104)
HUBBLESITE.org


f_552
4.274e-03
0.0036
0.0027
0.0018
0.0009
-4.930e-32

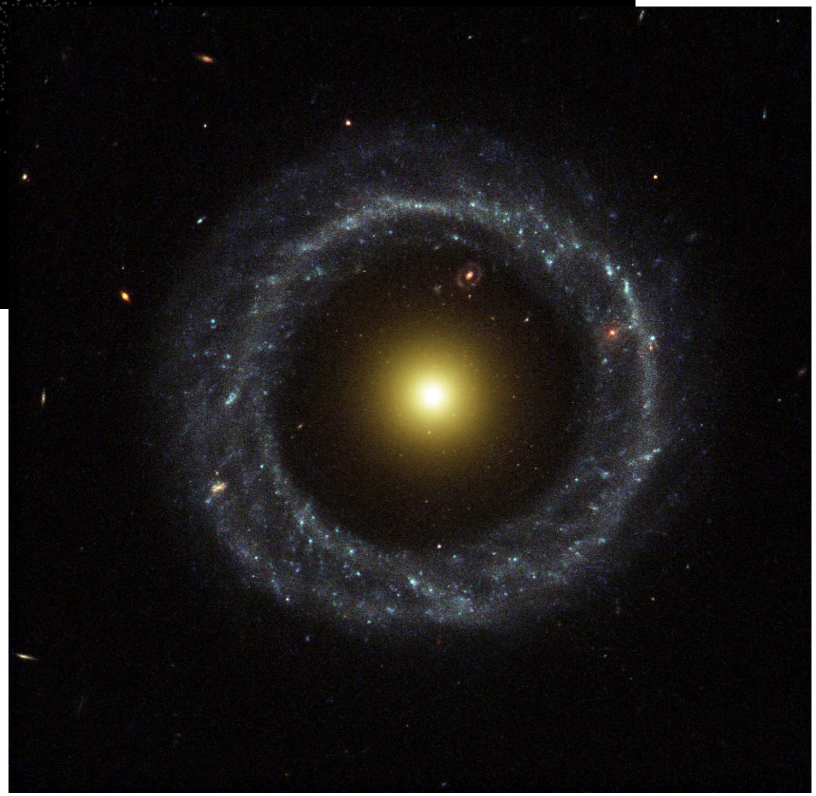


Numerical Solutions: Ring galaxy

$$\Phi(E,L) \propto C_{ring}\Phi_{ring}(E,L) + C_{center}\Phi_{center}(E,L)$$



Numerical Solutions: Ring galaxy



Ongoing work…

1. Implementing aposteriori error estimates and mesh refinement
2. Exploring / understanding limitations of code: relativistic solutions
3. Astrophysical relevance of solutions
4. Adding time-dependence: stability of solutions, collapse etc

Ongoing work…

1. Implementing aposteriori error estimates and mesh refinement
2. Exploring / understanding limitations of code: relativistic solutions
3. Astrophysical relevance of solutions
4. Adding time-dependence: stability of solutions, collapse etc

Thanks!