Well-Posed Boundary Conditions for the Incompressible Vorticity Equation Using a New High Order Mimetic Arakawa-Like Jacobian Differential Operator

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2-D Incompressible Vorticity Equation

Continuous model:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \mathbf{v}\cdot\nabla\zeta &= 0 \\ \nabla\cdot\mathbf{v} &= 0 \end{aligned} \quad \text{where} \quad \mathbf{v} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \tag{1}$$

ζ is the vorticity and ψ the streamfunction of 2-D incompressible flow
In this setting (1) is equivalent to the non-linear system:

$$\begin{cases} \zeta = \Delta\psi \\ \dfrac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0 \end{cases}$$

Jacobian Differential Operator: $J(a, b) = \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}$

Analytical Properties of the Jacobian Operator

Lemma

- *Jacobian operator is skew-symmetric by definition:* $J(a,b) = -J(b,a)$
- *Integral constraints following by Integration by parts*

$$\overline{J(a,b)} = \overline{aJ(a,b)} = \overline{bJ(a,b)} = 0, \qquad (2)$$

where $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f\; dxdy$, *a and b are periodic functions over* Ω

- **Enstrophy Conservation**:

$$\frac{1}{2}\overline{\frac{\partial \zeta^2}{\partial t}} = \overline{\zeta J(\psi,\zeta)} = 0$$

- **Kinetic Energy Conservation**:

$$\overline{\frac{\partial}{\partial t}\left(\frac{1}{2}\nabla\psi\right)^2} = \overline{\psi\frac{\partial \zeta}{\partial t}} = \overline{\psi J(\psi,\zeta)} = 0$$

The Arakawa's approach using Jacobian formulation

Arakawa (1966):
"[...] if we can find a finite difference scheme which has constraints analogous to the integral constraints of the differential form, the solution will not show the false "noodling" following by computational instability"

- defining a linear combination of three different consistent Jacobians, Arakawa proved the result for second order central finite difference scheme and periodic problems

The new generation of the Arakawa's approach

1. We replicate the same result of Arakawa for periodic problems using arbitrary high order Summation-By-Parts (SBP) approximations

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1. We replicate the same result of Arakawa for periodic problems using arbitrary high order Summation-By-Parts (SBP) approximations
2. We extend the work to completely general problems imposing well-posed boundary conditions weakly with Simultaneous-Approximation-Term (SAT) technique

SBP semi-discretization of partial derivatives

SBP operators: $f_{x,y} \approx D_{x,y}\mathbf{f} = P_{x,y}^{-1}Q_{x,y}\mathbf{f}$ in 1D

$P_{x,y} > 0$ diagonal matrices s.t. they defines a quadrature rule

$Q_{x,y}$ are periodic operators satisfying $Q_{x,y} + Q_{x,y}^T = 0$

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$$\partial_x \mathbf{f} = (P_x^{-1} Q_x \otimes I_y)\mathbf{f} = diag((P_x^{-1} Q_x \otimes I_y)\mathbf{f})\mathbf{1}$$

$$\partial_y \mathbf{f} = (I_x \otimes P_y^{-1} Q_y)\mathbf{f} = diag((I_x \otimes P_y^{-1} Q_y)\mathbf{f})\mathbf{1}$$

- **Computational 2-D grid:** $\begin{cases} x_i, \; i \in 0, 1, 2, ..., N \\ y_j, \; j \in 0, 1, 2, ..., M \end{cases}$
- $\mathbf{f} = (f_{11}, ..., f_{1M}, f_{21} ..., f_{2M}, ..., f_{N1}, ..., f_{NM})$ vector of dim NM
- $\otimes$ is the **Kronecker product** of two matrices
- $diag(a) = \begin{bmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & ... & 0 & a_N \end{bmatrix}$ and $\mathbf{1} = (1, \cdots, 1)^T$

High order minetic SBP semi-discretization

Consider three consistent Jacobian discretizations:

$$
\begin{aligned}
J_1 &= \left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right) \\
&\approx \Big\{\left[diag((P_x^{-1}Q_x \otimes I_y)\psi)diag((I_x \otimes P_y^{-1}Q_y)\zeta)\right]\mathbf{1} \\
&\quad - \left[diag((I_x \otimes P_y^{-1}Q_y)\psi)diag((P_x^{-1}Q_x \otimes I_y)\zeta)\right]\mathbf{1}\Big\} \\
J_2 &= \left(\frac{\partial}{\partial x}\left(\psi\frac{\partial\zeta}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi\frac{\partial\zeta}{\partial x}\right)\right) \\
&\approx \Big\{(P_x^{-1}Q_x \otimes I_y)\left[diag(\psi)diag((I_x \otimes P_y^{-1}Q_y)\zeta)\right]\mathbf{1} \\
&\quad - (I_x \otimes P_y^{-1}Q_y)\left[diag(\psi)diag((P_x^{-1}Q_x \otimes I_y)\zeta)\right]\mathbf{1}\Big\} \qquad (3) \\
J_3 &= \left(-\frac{\partial}{\partial x}\left(\zeta\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\zeta\frac{\partial\psi}{\partial x}\right)\right) \\
&\approx \Big\{-(I_x \otimes P_y^{-1}Q_y)\left[diag(\zeta)diag((P_x^{-1}Q_x \otimes I_y)\psi)\right]\mathbf{1} \qquad (4) \\
&\quad + (P_x^{-1}Q_x \otimes I_y)\left[diag(\zeta)diag((I_x \otimes P_y^{-1}Q_y)\psi)\right]\mathbf{1}\Big\}.
\end{aligned}
$$

Note that the continuous J_1, J_2, J_3 are equivalent expressions

High order minetic SBP semi-discretization

The discrete J_1, J_2, J_3 have different properties:

- J_1 is skew-symmetric
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Our result is

Theorem

The linear combination

$$J^* = \frac{1}{3}[J_1 + J_2 + J_3] \tag{5}$$

is skew-symmetric, conserves enstrophy and kinetic energy
Stability follows directly by conservation of enstrophy [a].

[a] Chiara Sorgentone, Cristina La Cognata, Jan Nordström, *A New High Order Energy and Enstrophy Conserving Arakawa-like Jacobian Differential Operator*. Accepted in Journal of Computational Physics

Finally

- **The SBP formulation allows arbitrary high order accurate J^***

Method of Manufactured Solution (MMS)

Analytical stream function:

$$\psi(x, y, t) = K\{sin[2\pi(l_1 x - t)] + cos[2\pi(l_2 y - t)]\}$$

K is a rescaling factor, $l_1 \neq l_2$ are constant.
Derive the manufactured vorticity from ψ

$$\zeta(x, y, t) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

and the forcing term:

$$f(x, y, t) = \zeta_t + J(\psi, \zeta)$$

Finally we solve:

$$\zeta_t + J(\psi, \zeta) = f$$

Accuracy and Efficiency

Accuracy

- $\Delta t = C(h)^{p/4}$
- 4th order Runge-Kutta explicit time-integrator

N	SBP 2th Err	SBP 2th p	SBP 4th Err	SBP 4th p	SBP 6th Err	SBP 6th p	SBP 8th Err	SBP 8th p
40	$1.10\ 10^{-2}$	1.97	$4.33\ 10^{-4}$	3.95	$1.84\ 10^{-5}$	5.88	$8.52\ 10^{-7}$	7.83
50	$7.50\ 10^{-3}$	1.94	$1.79\ 10^{-4}$	3.94	$4.90\ 10^{-6}$	5.93	$1.46\ 10^{-7}$	7.90
60	$5.34\ 10^{-3}$	2.03	$8.67\ 10^{-5}$	3.99	$1.65\ 10^{-6}$	5.95	$3.43\ 10^{-8}$	7.93
70	$3.92\ 10^{-3}$	2.00	$4.69\ 10^{-5}$	3.98	$6.59\ 10^{-7}$	5.96	$1.00\ 10^{-8}$	7.95

Efficiency

Comparison between the Arakawa (or second order SBP) scheme and high order approximations of J^* using SBP operators 4th, 6th and 8th order

- Fixed time step $\Delta t = 10^{-3}$ and Final time $T = 0.1$

	Arakawa	SBP 4th	SBP 6th	SBP 8th
Error	$4.83 \cdot 10^{-4}$	$4.34 \cdot 10^{-4}$	$4.70 \cdot 10^{-4}$	$4.08 \cdot 10^{-4}$
CPU	1215.614 s	2.091 s	0.287 s	0.139 s
N	200	40	23	18

Weak Boundary conditions - Continuous case

Consider the bounded domain Ω and the dissipative vorticity equation in Arakawa's formulation

$$\xi_t + \frac{1}{3}\left[J_1(\psi,\xi) + J_2(\psi,\xi) + J_3(\psi,\xi)\right] = \epsilon\Delta\xi,$$

The energy method gives:

$$\|\xi\|_t^2 + 2\|\nabla\xi\|^2 = -\frac{2}{3}\int_{\partial\Omega}\left[\xi^2\nabla^\perp\psi\cdot n - \xi\psi\,\nabla^\perp\xi\cdot n\right] + 2\epsilon\int_{\partial\Omega}\xi\,\partial_n\xi$$

We want to bound the RHS to get an energy estimate

Continuous boundary conditions

$$T(x,y) = \xi(\nabla^{\perp}\psi \cdot n) - \psi(\nabla^{\perp} \cdot \xi), \qquad (x,y) \in \partial\Omega.$$

A boundary condition that bounds the energy is

$$BC = -\frac{2}{3}\left[\frac{\xi T - |\xi T|}{2|\xi|}\right] - \epsilon\frac{\partial \xi}{\partial n} = 0.$$

BC changes expression depending on the sign of ξT, namely

$$\mathrm{BC} = \begin{cases} -\frac{2}{3}T - \epsilon\frac{\partial \xi}{\partial_n} = 0, & \text{if } \xi T < 0, \\ -\epsilon\frac{\partial \xi}{\partial n} = 0, & \text{if } \xi T > 0, \end{cases}$$

Continuous energy estimate

To bound the energy we add the null penalty term $-\int_{\partial\Omega} 2\sigma\xi \cdot \mathrm{BC}$

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$$\|\xi\|_t^2 + 2\epsilon\|\nabla\xi\|^2 = -\int_{\partial\Omega} \left\{ \frac{2}{3}\left[\xi T + 2\sigma\frac{\xi T - |\xi T|}{2|\xi|}\right] - (1+\sigma)2\epsilon\xi\frac{\partial\xi}{\partial n} \right\}\, ds$$

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and with the choice $\sigma = -1$, we get

$$\|\xi\|_t^2 + 2\xi\epsilon\|\nabla\xi\|^2 = -\int_{\partial\Omega} \left\{ \frac{2}{3}\left[\xi T - 2\xi\left(\frac{\xi T - |\xi T|}{2|\xi|}\right)\right] \right\}.$$

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defining $\partial\Omega_i^+$ the intervals where ξT is positive and $\partial\Omega_j^-$ where it is negative

$$\|\xi\|_t^2 + 2\epsilon\|\nabla\xi\|^2 = \sum_i \int_{\partial\Omega_i^+} \left[-\frac{2}{3}\xi T\right] + \sum_j \int_{\partial\Omega_j^-} \left[\frac{2}{3}\xi T\right] \leq 0,$$

The semi-discrete energy estimate

Consider the SBP semi-discretization

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \frac{1}{3}\left[\mathbf{J}_1(\boldsymbol{\psi},\boldsymbol{\xi}) + \mathbf{J}_2(\boldsymbol{\psi},\boldsymbol{\xi}) + \mathbf{J}_3(\boldsymbol{\psi},\boldsymbol{\xi})\right]$$
$$= \epsilon\left[(P_x^{-1}Q_x \otimes I_y)^2 + (I_x \otimes P_y^{-1}Q_y)^2\right]\boldsymbol{\xi}$$

We apply the discrete energy method by multiply from the left by

$$\boldsymbol{\xi}^T(P_x \otimes P_y)$$

and mimic summation by part rule by using

$$Q_{x,y} = -Q_{x,y}^T + B_{x,y}$$

where

$$B_{x,y} = \begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix} \quad \text{are boundary operators}$$

The semi-discrete energy

$$\frac{\partial}{\partial t}\|\xi^2\|^2_{(P_x\otimes P_y)} + 2\epsilon(\|(P_x^{-1}Q_x \otimes I_y\xi)^2\|^2_{(P_x\otimes P_y)} + \|(I_x \otimes P_y^{-1}B_y\xi)^2\|^2_{(P_x\otimes P_y)})$$

$$= \frac{2}{3}\mathbf{1}^T(P_x \otimes P_y)\left\{(P_x^{-1}B_x \otimes I_y)\left[diag(\xi)diag(\psi)diag(I_x \otimes P_y^{-1}Q_y\xi)\right]\mathbf{1}\right.$$
$$- (I_x \otimes P_y^{-1}B_y)\left[diag(\xi)diag(\psi)diag(P_x^{-1}Q_x \otimes I_y\xi)\right]\mathbf{1}$$
$$+ (I_x \otimes P_y^{-1}B_y)\left[diag(\xi)diag(\xi)diag(P_x^{-1}Q_x \otimes I_y\psi)\right]\mathbf{1}$$
$$\left.- (P_x^{-1}B_x \otimes I_y)\left[diag(\xi)diag(\xi)diag(I_x \otimes P_y^{-1}Q_y\psi)\right]\mathbf{1}\right\}$$

$$+ 2\epsilon\mathbf{1}^T(P_x \otimes P_y)\left\{(P_x^{-1}B_x \otimes I_y)\left[diag(\xi)diag(P_x^{-1}Q_x \otimes I_y\xi)\right]\right.$$
$$\left.+(I_x \otimes P_y^{-1}B_y)\left[diag(\xi)diag(I_x \otimes P_y^{-1}Q_y\xi)\right]\right\}.$$

Semi-discrete boundary conditions

The discrete analogous of T

$$
\begin{aligned}
T_i = \Big\{ & (P_x^{-1} B_x \otimes I_y) \left[diag(\boldsymbol{\xi}) diag(\boldsymbol{\psi}) diag(I_x \otimes P_y^{-1} Q_y \boldsymbol{\xi}) \right] \mathbf{1} \\
& - (I_x \otimes P_y^{-1} B_y) \left[diag(\boldsymbol{\xi}) diag(\boldsymbol{\psi}) diag(P_x^{-1} Q_x \otimes I_y \boldsymbol{\xi}) \right] \mathbf{1} \\
& + (I_x \otimes P_y^{-1} B_y) \left[diag(\boldsymbol{\xi}) diag(\boldsymbol{\xi}) diag(P_x^{-1} Q_x \otimes I_y \boldsymbol{\psi}) \right] \mathbf{1} \\
& - \, (P_x^{-1} B_x \otimes I_y) \left[diag(\boldsymbol{\xi}) diag(\boldsymbol{\xi}) diag(I_x \otimes P_y^{-1} Q_y \boldsymbol{\psi}) \right] \mathbf{1} \Big\}_i
\end{aligned}
$$

and the SAT vector of penalties

$$
\begin{aligned}
&\mathrm{SAT}_i = -2\tau \left\{ \frac{2}{3} \frac{\xi_i T_i - |\xi_i T_i|}{2|\xi_i|} \right\} \\
&+ \epsilon \, \left[(P_x^{-1} B_x \otimes I_y) diag(P_x^{-1} Q_x \otimes I_y \boldsymbol{\xi}) + (I_x \otimes P_y^{-1} B_y) diag(I_x \otimes P_y^{-1} Q_y \boldsymbol{\xi}) \right]_i
\end{aligned}
$$

and $\mathrm{SAT}_i = 0$ when $\xi_i = 0$.

The semi-discrete energy estimate

To bound the discrete energy we add the SAT vector to the discrete energy and imposing $\tau = -1$,

$$\frac{\partial}{\partial t}\|\boldsymbol{\xi}^2\|^2_{(P_x\otimes P_y)} + 2\epsilon(\|(P_x^{-1}Q_x \otimes I_y\boldsymbol{\xi})^2\|^2_{(P_x\otimes P_y)} + \|(I_x \otimes P_y^{-1}B_y\boldsymbol{\xi})^2\|^2_{(P_x\otimes P_y)})$$
$$= -\frac{2}{3}\sum_{i\in D^+}(P_x \otimes P_y)_{ii}\xi_i T_i + \frac{2}{3}\sum_{j\in D^-}(P_x \otimes P_y)_{jj}\xi_j T_j \leq 0.$$

we get a discrete energy estimate similar to the continuous one which ensures stability

D^+ the set of indices of $\mathbf{T}$ such that $\xi_i T_i > 0$ and D^- the set of indices such that $\xi_i T_i < 0$

Summary and Conclusions

1. The SBP formulation allows arbitrary high order accurate approximation of the Arakawa's like Jacobian J^*
2. For periodic problems, the SBP-J^* mimics the analytical properties of the continuous Jacobian
3. Well-posed boundary conditions for the dissipative vorticity equation are derived on general domains
4. SAT technique is used to weakly imposed boundary conditions to the approximation and make it stable

Thank you!