Coupling Requirements for Well Posed and Stable Multi-Physics Problems

By

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What can be coupled?

- Hyperbolic eqs coupled to hyperbolic eqs.
- Parabolic eqs coupled to parabolic eqs.
- Hyperbolic eqs coupled to parabolic eqs will create indefinite boundary terms and ruin energy stability.
- Coupling of m equations to n equations, m=n rare.
- General principles guiding these coupling procedures?
- What can be coupled?
- How about well-posedness?
- Conservation? What does that mean in the general case?
- The dual/adjoint problem?

A model problem

Interface

$$u_t + Au_x = 0 \quad \Big| \quad v_t + Bv_x = 0$$

$$Cu = Dv$$

With A, B given, what is C, D?

A simplified model problem

We will consider the following system,

$$u_t + Au_x = 0, \qquad -1 \le x \le 0, \qquad t > 0, \tag{1}$$
$$u(x,0) = f(x),$$

and the scalar equation

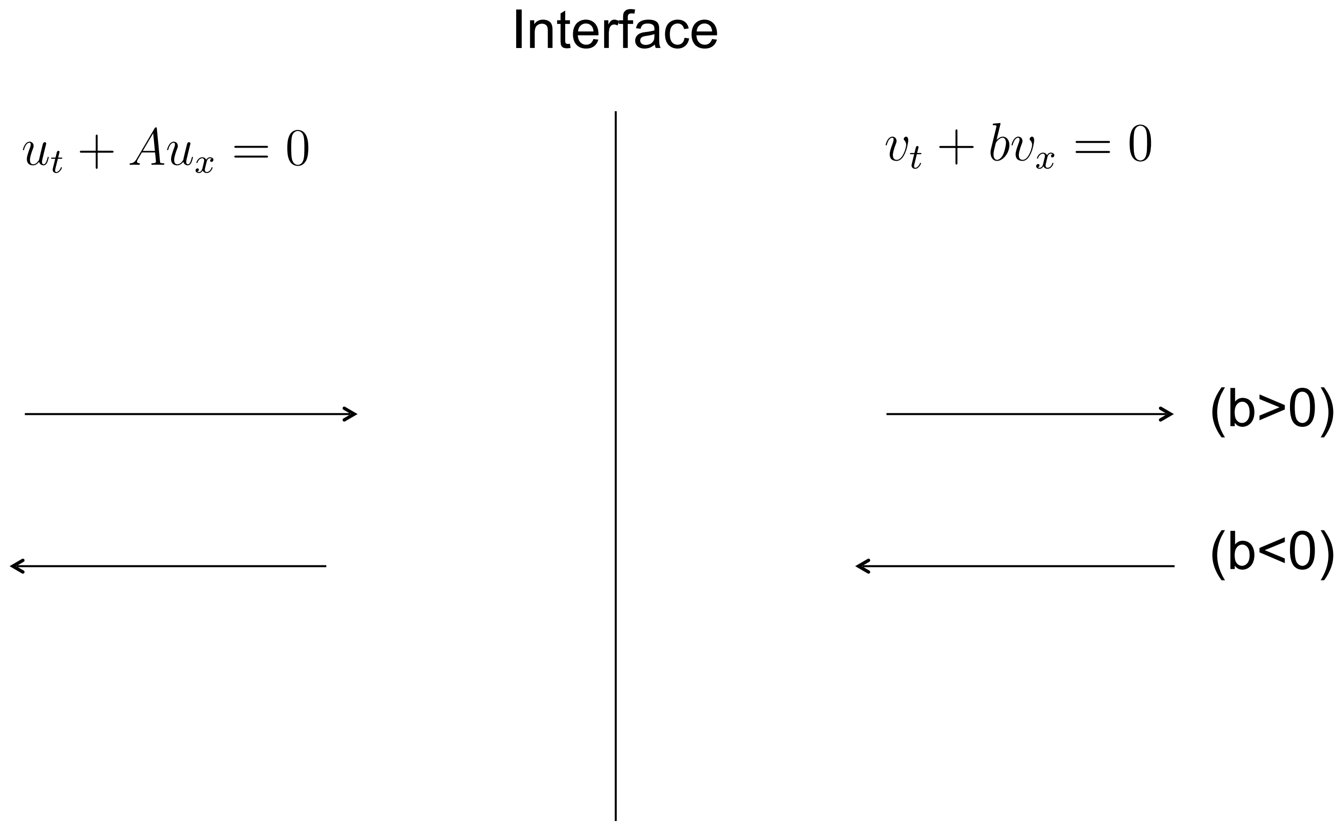
$$v_t + Bv_x = 0, \qquad 0 \le x \le 1, \qquad t > 0, \tag{2}$$
$$v(x,0) = g(x).$$

$u = (u_1, u_2)^T$ a vector of unknowns,
$f(x) = (f_1(x), f_2(x))^T$ a vector of given data

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, a > 0.$$

Two boundary/interface conditions are needed for the system (1) while equation (2) needs one boundary/interface condition.

Wave propagation



(b>0) require two interface conditions

(b<0) require one interface condition

Energy

Applying the energy method to both equations

$$\frac{d}{dt}(\|u\|_2^2 + \alpha\|v\|_2^2) = -u^T Au|_{x=0} + \alpha b v^2|_{x=0} = w^T E w, \quad (3)$$

α a positive free weight, $w = [u_1, u_2, v]^T$

$$E = \begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ 0 & 0 & \alpha b \end{bmatrix}.$$

The eigenvalues of E are $\{a, -a, \alpha b\}$.
If $b < 0$, $\Rightarrow$ one of the eigenvalues is positive $\Rightarrow$ one condition at $x = 0$
If $b > 0$, $\Rightarrow$ two of the eigenvalues are positive $\Rightarrow$ two conditions at $x = 0$

The boundary matrix

In order to couple the problems we need at least one accuracy condition. Let

$$v = C^T u, \quad C = [c_1, c_2]^T. \tag{4}$$

The relation (4) inserted in equation (3) leads to

$$\frac{d}{dt}(\|u\|_2^2 + \alpha\|v\|_2^2) = u^T(0,t)Du(0,t),$$

$D = (\alpha bCC^T - A)$. The characteristic polynomial related to the eigenvalues λ of D is

$$\lambda^2 - \alpha b(c_1^2 + c_2^2)\lambda + 2\alpha abc_1c_2 - a^2.$$

$$2s_1 = -\alpha b(c_1^2 + c_2^2) \text{ and } s_2 = 2\alpha abc_1c_2 - a^2$$

$$\Downarrow$$

$$\lambda_{1,2} = -s_1 \pm \sqrt{s_1^2 - s_2}$$

Eigenvalues and coupling conditions

$$\lambda_{1,2} = -s_1 \pm \sqrt{s_1^2 - s_2}. \qquad 2s_1 = -\alpha b(c_1^2 + c_2^2) \text{ and } s_2 = 2\alpha a b c_1 c_2 - a^2$$

$b < 0 \Rightarrow s_1 > 0$

$c_1 c_2 < a/(2\alpha b) \Rightarrow s_2 > 0$ Only negative eigenvalues , OK, But need $c_1 c_2 < 0$

$c_1 c_2 > a/(2\alpha b) \Rightarrow s_2 < 0$ One positive eigenvalue , Not OK

$b > 0 \Rightarrow s_1 < 0$

$c_1 c_2 < a/(2\alpha b) \Rightarrow s_2 < 0$ One positive eigenvalue , OK, But need a boundary condition

$c_1 c_2 > a/(2\alpha b) \Rightarrow s_2 > 0$ Two positive eigenvalues , Not OK but can always be avoided by choosing a small α

Tentative conclusions

$$v = C^T u, \quad C = [c_1, c_2]^T$$

The accuracy condition leads to a well posed problem only for certain c_1 and c_2

$b < 0 :$ An energy estimate (negative semi-definite eigenvalues of D) if c_1 and c_2 have opposite signs. Only the interface condition required.

$b > 0 :$ An energy estimate for all values of c_1 and c_2. One additional condition, (a boundary condition) is required.

The numerical approximation

First, we consider $b < 0$. The semi-discrete SBP-SAT formulations of (1) and (2) are,

$$\mathbf{u}_t + (D_u \otimes A)\mathbf{u} = (P_u^{-1} E_N^u \otimes \Sigma)(C^T \tilde{u}_N - v_0) e_N^u,$$
$$\mathbf{v}_t + b D_v \mathbf{v} = P_v^{-1} \sigma (v_0 - C^T \tilde{u}_N) e_0^v. \qquad (5)$$

$D_{u,v} = P_{u,v}^{-1} Q_{u,v}$ are the difference operators, $P_{u,v}$ are positive definite matrices and $Q_{u,v}$ satisfy $Q_{u,v} + Q_{u,v}^T = diag[-1, \cdots, 1]$ and

$$\mathbf{u} = (u_{10}, u_{20}, \cdots u_{1N}, u_{2N}), \qquad \mathbf{v} = (v_0, \cdots, v_M).$$

The vectors $e_N^u = (0, \cdots, 0, 1, 1)^T$ and $e_0^v = (1, \cdots, 0)^T$ are $2N \times 1$ and $M \times 1$, respectively. $E_N^u = diag[0, \cdots, 1]$ and $E_0^v = diag[1, \cdots, 0]$ are $N \times N$ and $M \times M$, respectively. The penalty matrix Σ is given by

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{bmatrix},$$

σ is penalty parameter, and also, $\tilde{u}_N = [u_{1N}, u_{2N}]^T$.

Next, consider $b > 0$. The semi-discrete SBP-SAT formulations of (1) and (2) are

$$\begin{aligned}\mathbf{u}_t + (D_u \otimes A)\mathbf{u} &= (P_u^{-1} E_N^u \otimes \Sigma)(C^T \tilde{u}_N - v_0) e_N^u + (P_u^{-1} E_N^u \otimes \Xi \tilde{H})(\mathbf{u} - e_N^u \otimes \tilde{h}),\\ \mathbf{v}_t + bD_v \mathbf{v} &= P_v^{-1} \sigma (v_0 - C^T \tilde{u}_N) e_0^v.\end{aligned} \tag{6}$$

where the penalty matrix Ξ and $\tilde{H}$ are given by

$$\Xi = \begin{bmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 1 & -R_r \\ 0 & 0 \end{bmatrix} Y^T.$$

The boundary data $\tilde{h}$ is defined as $\tilde{h} = [0, h]^T$.

Stability

First we consider $b < 0$. The discrete energy method is applied to (5) yields

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u \otimes I} + \alpha_d \|\mathbf{v}\|^2_{P_v}) = -\tilde{u}_N^T A \tilde{u}_N + \alpha_d b v_0^2 + 2\tilde{u}_N^T \Sigma H + 2\alpha_d \sigma v_0 (v_0 - C^T \tilde{u}_N). \qquad (7)$$

α_d is a positive weight and $H = [C^T\tilde{u}_N - v_0, C^T\tilde{u}_N - v_0]^T$.
In order to mimic the continuous case, we choose $\Sigma = \alpha b/2 \begin{bmatrix} 0 & C \end{bmatrix}$. By inserting that into (7) we get

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u \otimes I} + \alpha_d \|\mathbf{v}\|^2_{P_v}) = \tilde{u}_N^T D \tilde{u}_N + \alpha_d v_0^2 (b + 2\sigma) - \sigma v_0 C^T \tilde{u}_N (\alpha b + 2\alpha_d \sigma). \qquad (8)$$

If we choose $\sigma = -\alpha b/2\alpha_d$, for $\alpha_d \leq \alpha$ the right-hand side of (8) will be bounded due to the continuous result above.

Next, we consider $b > 0$ and let $h(t) = 0$. Multiplying (6) by $\mathbf{u}^T(P_u \otimes I)$ and $\mathbf{v}^T P_v$ leads to

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u \otimes I} + \alpha_d \|\mathbf{v}\|^2_{P_v}) \leq \tilde{u}_N^T (D + \Xi\tilde{H} + (\Xi\tilde{H})^T)\tilde{u}_N,$$

where we have chosen Σ and σ as for the case $b < 0$.

By choosing

$$\Xi = Y \begin{bmatrix} -\lambda^+ & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{d}{dt}(\|\mathbf{u}\|^2_{P_u \otimes I} + \alpha_d \|\mathbf{v}\|^2_{P_v}) \leq 0,$$

for $\alpha_d \geq \alpha$.

Accuracy

The analytical solution

$$u_1(x,t) = u_2(x,t) = \cos(2\pi(x-t))v(x,t) = \sin(3\pi(x-bt))$$

The rate of convergence

$$q_u = \ln\left(\frac{\|(\mathbf{u}_1^{N_1},\mathbf{u}_2^{N_1}) - (u_1,u_2)\|_{P_u\otimes I}}{\|(\mathbf{u}_1^{N_2},\mathbf{u}_2^{N_2}) - (u_1,u_2)\|_{P_u\otimes I}}\right) / \ln\left(\frac{N_1}{N_2}\right), q_v = \ln\left(\frac{\|\mathbf{v}^{N_1} - v\|_{P_v}}{\|\mathbf{v}^{N_2} - v\|_{P_v}}\right) / \ln\left(\frac{N_1}{N_2}\right)$$

	$SBP21$		$SBP42$		$SBP63$		$SBP84$	
N	*error*	*rate*	*error*	*rate*	*error*	*rate*	*error*	*rate*
20	2e-2	-	2e-3	-	2e-3	-	1e-3	-
40	6e-3	1.877	3e-4	3.006	1e-4	4.035	3e-5	5.376
80	1e-3	2.046	3e-5	3.242	8e-6	4.224	8e-7	5.392
160	4e-4	1.985	3e-6	3.052	4e-7	4.470	2e-8	5.113
320	1e-4	2.004	4e-7	3.021	2e-8	4.375	6e-10	5.091
640	2e-5	1.998	6e-8	3.013	1e-9	4.077	2e-11	5.047

Table 1: error and rate q_u for $b < 0$.

Summary and future work

- We have discussed/questioned whether there are general principles governing coupled problems.
- As an intial study, a simple model problem was investigated.
- It was shown that the coupling (or accuracy) conditions had to satisfy certain conditions, otherwise well-posedness could not be obtained.
- This implies that some problems can be coupled, and others not.
- Stability and accuracy followed immidiately from the well-posedness results using the SBP-SAT technique.
- Future work: investigation regarding conservation and duality.
- Future work: the relation between well-posednes, stability, conservation and duality.

Thank you for listening !

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