Uncertainty Quantification for High Frequency Waves

Gabriela Malenová

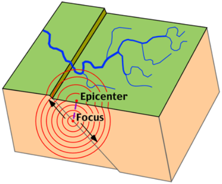
Royal Institute of Technology KTH

BIT Circus 2015, Umeå, August 27

Jointly with Olof Runborg, Mohammad Motamed, Raul Tempone

Problem statement

- Propagation of high-frequency waves with uncertain parameters.
- e.g. earthquakes: uncertain medium and source location



Simplified model: scalar wave equation with

1. Highly oscillatory initial data.
2. Uncertainty (initial data and/or model parameters).

Cauchy problem for the scalar wave equation

$$
\begin{aligned}
u^{\varepsilon}_{tt}(t,\mathbf{x}) &= c(\mathbf{x})^2 \Delta u^{\varepsilon}(t,\mathbf{x}), && (t,\mathbf{x}) \in \mathbb{R}^+ \times \mathbb{R}^n,\\
u^{\varepsilon}(0,\mathbf{x}) &= A_0(\mathbf{x}) e^{i\Phi_0(\mathbf{x})/\varepsilon}, && \\
u^{\varepsilon}_{t}(0,\mathbf{x}) &= \frac{1}{\varepsilon} B_0(\mathbf{x}) e^{i\Phi_0(\mathbf{x})/\varepsilon}, && t=0,\ \mathbf{x} \in \mathbb{R}^n,
\end{aligned}
$$

with c wave speed, Φ_0 initial phase, $\varepsilon \ll 1$ wavelength and A_0, B_0 amplitude parameters.

Cauchy problem for the scalar wave equation

$$
\begin{aligned}
u^{\varepsilon}_{tt}(t,\mathbf{x}) &= c(\mathbf{x},\mathbf{y})^2\Delta u^{\varepsilon}(t,\mathbf{x}), \qquad && (t,\mathbf{x},\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Gamma, \\
u^{\varepsilon}(0,\mathbf{x}) &= A_0(\mathbf{x})e^{i\Phi_0(\mathbf{x})/\varepsilon}, && \\
u^{\varepsilon}_{t}(0,\mathbf{x}) &= \frac{1}{\varepsilon}B_0(\mathbf{x})e^{i\Phi_0(\mathbf{x})/\varepsilon}, && t=0,\ (\mathbf{x}) \in \mathbb{R}^n,
\end{aligned}
$$

with c wave speed, Φ_0 initial phase, $\varepsilon \ll 1$ wavelength and A_0, B_0 amplitude parameters.

Sources of uncertainty:

- Wave speed $c = c(\mathbf{x},\mathbf{y})$, $\quad \mathbf{y} \in \Gamma \subset \mathbb{R}^N$ random vector,

Cauchy problem for the scalar wave equation

$$
\begin{aligned}
u^\varepsilon_{tt}(t,\mathbf{x}) &= c(\mathbf{x},\mathbf{y})^2 \Delta u^\varepsilon(t,\mathbf{x}), && (t,\mathbf{x},\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Gamma,\\
u^\varepsilon(0,\mathbf{x}) &= A_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon}, &&\\
u^\varepsilon_t(0,\mathbf{x}) &= \frac{1}{\varepsilon} B_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon}, && t=0,\ (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \Gamma,
\end{aligned}
$$

with c wave speed, Φ_0 initial phase, $\varepsilon \ll 1$ wavelength and A_0, B_0 amplitude parameters.

Sources of uncertainty:

- Wave speed $c = c(\mathbf{x},\mathbf{y})$, $\quad \mathbf{y} \in \Gamma \subset \mathbb{R}^N$ random vector,
- Initial data A_0, B_0, Φ_0.

Cauchy problem for the scalar wave equation

$$
\begin{aligned}
u^\varepsilon_{tt}(t,\mathbf{x},\mathbf{y}) &= c(\mathbf{x},\mathbf{y})^2 \Delta u^\varepsilon(t,\mathbf{x},\mathbf{y}), && (t,\mathbf{x},\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Gamma,\\
u^\varepsilon(0,\mathbf{x},\mathbf{y}) &= A_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon},\\
u^\varepsilon_t(0,\mathbf{x},\mathbf{y}) &= \frac{1}{\varepsilon} B_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon}, && t = 0,\ (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \Gamma,
\end{aligned}
$$

with c wave speed, Φ_0 initial phase, $\varepsilon \ll 1$ wavelength and A_0, B_0 amplitude parameters.

Sources of uncertainty:

- Wave speed $c = c(\mathbf{x},\mathbf{y})$, $\quad \mathbf{y} \in \Gamma \subset \mathbb{R}^N$ random vector,
- Initial data A_0, B_0, Φ_0.

$\Rightarrow$ uncertainty in $u^\varepsilon = u^\varepsilon(t,\mathbf{x},\mathbf{y})$.

Goals

- Consider quantity of interest

$$\mathcal{Q}^{\varepsilon}(\mathbf{y}) = \int_{\mathbb{R}^n} |u^{\varepsilon}(T, \mathbf{x}, \mathbf{y})|^2 \, \psi(\mathbf{x}) \, d\mathbf{x}, \quad \psi \in C_c^{\infty}(\mathbb{R}^n).$$

where $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$.

Goals

- Consider quantity of interest

$$\mathcal{Q}^{\varepsilon}(\mathbf{y}) = \int_{\mathbb{R}^n} |u^{\varepsilon}(T, \mathbf{x}, \mathbf{y})|^2 \psi(\mathbf{x})\, d\mathbf{x}, \quad \psi \in C_c^{\infty}(\mathbb{R}^n).$$

  where $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$.

- Want to compute

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] = \int_{\Gamma} \mathcal{Q}^{\varepsilon}(\mathbf{y}) d\mathbf{y}.$$

Goals

- Consider quantity of interest

$$\mathcal{Q}^\varepsilon(\mathbf{y}) = \int_{\mathbb{R}^n} |u^\varepsilon(T, \mathbf{x}, \mathbf{y})|^2 \psi(\mathbf{x})\, d\mathbf{x}, \quad \psi \in C_c^\infty(\mathbb{R}^n).$$

  where $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$.

- Want to compute

$$\mathbb{E}[\mathcal{Q}^\varepsilon(\mathbf{y})] = \int_\Gamma \mathcal{Q}^\varepsilon(\mathbf{y}) d\mathbf{y}.$$

Proposed method:

1. Gaussian beam method.
2. Sparse stochastic collocation.

Layout

High frequency approximations

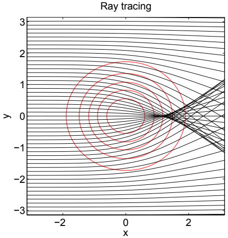
- Geometrical optics
- Gaussian beam method

Uncertainty quantification

- Stochastic collocation
- Stochastic regularity for high frequency waves

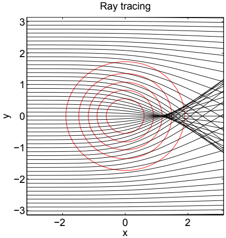
Numerical examples

Geometrical optics



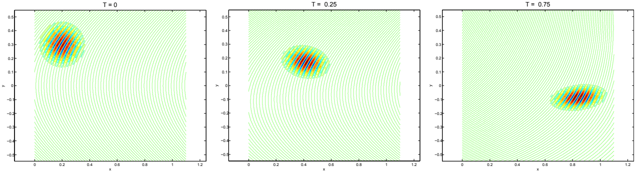
- For high frequencies $\varepsilon \ll 1$, computational cost of direct methods grows rapidly.
- Geometrical optics: approximation in the limit $\varepsilon \to 0$.
- GO breaks down at caustics.
- Remedy: Gaussian beam method.

Geometrical optics



- For high frequencies $\varepsilon \ll 1$, computational cost of direct methods grows rapidly.
- Geometrical optics: approximation in the limit $\varepsilon \to 0$.
- GO breaks down at caustics.
- Remedy: Gaussian beam method.

Gaussian beams



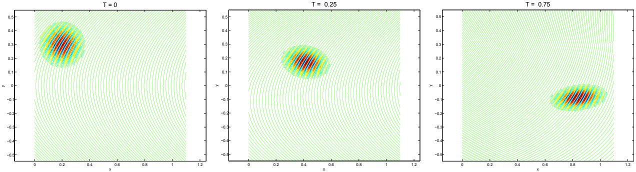
- High-frequency approximation using same Ansatz as GO

$$v^{\varepsilon}(t,\mathbf{x},\mathbf{y}) = A(t,\mathbf{x},\mathbf{y})e^{i\Phi(t,\mathbf{x},\mathbf{y})/\varepsilon}, \quad \mathbf{y} \in \Gamma \subset \mathbb{R}^N$$

with Φ and A Taylor expanded locally around GO ray $\mathbf{q}(t,\mathbf{y})$

$$A(t,\mathbf{x},\mathbf{y}) = a(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y}),\mathbf{y}), \qquad \Phi(t,\mathbf{x},\mathbf{y}) = \phi(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y}),\mathbf{y}).$$

Gaussian beams



- High-frequency approximation using same Ansatz as GO

$$v^\varepsilon(t,\mathbf{x},\mathbf{y}) = A(t,\mathbf{x},\mathbf{y})e^{i\Phi(t,\mathbf{x},\mathbf{y})/\varepsilon}, \quad \mathbf{y}\in\Gamma\subset\mathbb{R}^N$$

with Φ and A Taylor expanded locally around GO ray $\mathbf{q}(t,\mathbf{y})$

$$A(t,\mathbf{x},\mathbf{y}) = a(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y}),\mathbf{y}), \qquad \Phi(t,\mathbf{x},\mathbf{y}) = \phi(t,\mathbf{x}-\mathbf{q}(t,\mathbf{y}),\mathbf{y}).$$

- Full solutions on a small $\sim\sqrt{\varepsilon}$ neighborhood around the ray.
- Φ has positive imaginary part away from the ray.

Gaussian beams

First order beams:

$$a(t,\mathbf{x},\mathbf{y}) = a_0(t,\mathbf{y}), \quad \phi(t,\mathbf{x},\mathbf{y}) = \phi_0(t,\mathbf{y}) + \mathbf{x}\cdot\mathbf{p}(t,\mathbf{y}) + \frac{1}{2}\mathbf{x}\cdot M(t,\mathbf{y})\mathbf{x}.$$

- Require: $\Phi(t,\mathbf{x},\mathbf{y})$ solves eikonal equation to $O(|\mathbf{x}-\mathbf{q}(t,\mathbf{y})|^3)$ and $A(t,\mathbf{x},\mathbf{y})$ solves transport equation to $O(|\mathbf{x}-\mathbf{q}(t,\mathbf{y})|)$.

Gaussian beams

First order beams:

$$a(t, \mathbf{x}, \mathbf{y}) = a_0(t, \mathbf{y}), \quad \phi(t, \mathbf{x}, \mathbf{y}) = \phi_0(t, \mathbf{y}) + \mathbf{x} \cdot \mathbf{p}(t, \mathbf{y}) + \frac{1}{2} \mathbf{x} \cdot M(t, \mathbf{y}) \mathbf{x}.$$

- Require: $\Phi(t, \mathbf{x}, \mathbf{y})$ solves eikonal equation to $O(|\mathbf{x} - \mathbf{q}(t, \mathbf{y})|^3)$ and $A(t, \mathbf{x}, \mathbf{y})$ solves transport equation to $O(|\mathbf{x} - \mathbf{q}(t, \mathbf{y})|)$.
- We obtain set of ODEs for $\mathbf{q}, \mathbf{p}, \phi_0, M, a_0$.

Gaussian beams

First order beams:

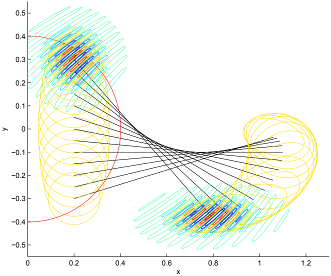
$$a(t,\mathbf{x},\mathbf{y}) = a_0(t,\mathbf{y}), \quad \phi(t,\mathbf{x},\mathbf{y}) = \phi_0(t,\mathbf{y}) + \mathbf{x}\cdot\mathbf{p}(t,\mathbf{y}) + \frac{1}{2}\mathbf{x}\cdot M(t,\mathbf{y})\mathbf{x}.$$

- Require: $\Phi(t,\mathbf{x},\mathbf{y})$ solves eikonal equation to $O(|\mathbf{x}-\mathbf{q}(t,\mathbf{y})|^3)$ and $A(t,\mathbf{x},\mathbf{y})$ solves transport equation to $O(|\mathbf{x}-\mathbf{q}(t,\mathbf{y})|)$.
- We obtain set of ODEs for $\mathbf{q},\mathbf{p},\phi_0,M,a_0$.
- Gaussian shape

$$|v^\varepsilon(t,\mathbf{x},\mathbf{y})| = a_0 \exp(-\mathrm{Im}(\Phi)/\varepsilon) = a_0 \exp\left(-\frac{1}{2\varepsilon}(\mathbf{x}-\mathbf{q}(t,\mathbf{y}))\cdot \mathrm{Im}(M)(\mathbf{x}-\mathbf{q}(t,\mathbf{y}))\right).$$

- $M = M^T$ and $\mathrm{Im}(M) > 0$ for all $t > 0$ if valid for initial data.

Gaussian beam superposition

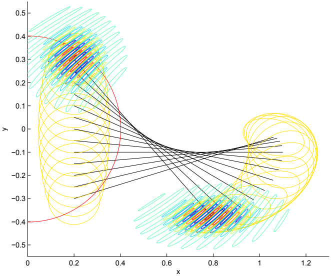


More general solutions $\Rightarrow$ superpositions of Gaussian beams:

$$u^{\varepsilon}_{GB}(t, \mathbf{x}, \mathbf{y}) = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} v^{\varepsilon}(t, \mathbf{x}, \mathbf{y}; \mathbf{z}) d\mathbf{z},$$

$K_0 \subset \mathbb{R}^n$ compact, $\mathbf{z} \in K_0$ is starting point.

Gaussian beam superposition



More general solutions $\Rightarrow$ superpositions of Gaussian beams:

$$u_{GB}^{\varepsilon}(t,\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi\varepsilon)^{n/2}} \int_{K_0} v^{\varepsilon}(t,\mathbf{x},\mathbf{y};\mathbf{z})d\mathbf{z},$$

$K_0 \subset \mathbb{R}^n$ compact, $\mathbf{z} \in K_0$ is starting point.

- By wave equation linearity, sum of solutions is also a solution
- Accuracy $\|u(t,\cdot) - u_{GB}(t,\cdot)\|_E \le C(t)\,\varepsilon^{1/2}$

Layout

Stochastic collocation

- Consider quantity of interest

$$\mathcal{Q}^\varepsilon(\mathbf{y}) = \int_{\mathbb{R}^n} |u^\varepsilon_{GB}(T, \mathbf{x}, \mathbf{y})|^2 \, \psi(\mathbf{x}) \, d\mathbf{x}, \quad \psi \in C_c^\infty(\mathbb{R}^n).$$

  where $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$ (random).
- Want to compute

$$\mathbb{E}[\mathcal{Q}^\varepsilon(\mathbf{y})] = \int_\Gamma \mathcal{Q}^\varepsilon(\mathbf{y}) d\mathbf{y}.$$

Stochastic collocation

- Consider quantity of interest

$$\mathcal{Q}^{\varepsilon}(\mathbf{y}) = \int_{\mathbb{R}^n} |u^{\varepsilon}_{GB}(T, \mathbf{x}, \mathbf{y})|^2 \, \psi(\mathbf{x}) \, d\mathbf{x}, \quad \psi \in C_c^{\infty}(\mathbb{R}^n).$$

  where $\mathbf{y} \in \Gamma \subset \mathbb{R}^N$ (random).
- Want to compute

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] = \int_{\Gamma} \mathcal{Q}^{\varepsilon}(\mathbf{y}) d\mathbf{y}.$$

- Approximated by

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] \approx \sum_{k=1}^{\eta} \alpha_k \mathcal{Q}^{\varepsilon}(\mathbf{y}^{(k)}),$$

  where α weights associated to the points used.
- NOTE: one full solve of high-frequency problem needed for each $\mathbf{y}$ value of $\mathcal{Q}(\mathbf{y})$.

Sparse grid quadrature

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] \approx \sum_{k=1}^{\eta} \alpha_k \mathcal{Q}^{\varepsilon}(\mathbf{y}^{(k)}),$$

- Key point: choice of collocation point set $\{\mathbf{y}^{(k)}\}_{k=1}^{\eta} \in \Gamma$.

Sparse grid quadrature

$$\mathbb{E}[\mathcal{Q}^\varepsilon(\mathbf{y})] \approx \sum_{k=1}^{\eta} \alpha_k \mathcal{Q}^\varepsilon(\mathbf{y}^{(k)}),$$

- Key point: choice of collocation point set $\{\mathbf{y}^{(k)}\}_{k=1}^{\eta} \in \Gamma$.
- Full grids expensive for $N \gg 1$ (curse of dimensionality).
- Standard quadrature slow when N large.

Sparse grid quadrature

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] \approx \sum_{k=1}^{\eta} \alpha_k \mathcal{Q}^{\varepsilon}(\mathbf{y}^{(k)}),$$

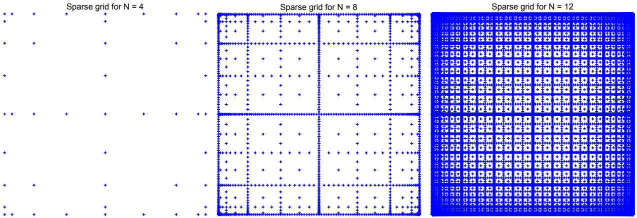
- Key point: choice of collocation point set $\{\mathbf{y}^{(k)}\}_{k=1}^{\eta} \in \Gamma$.
- Full grids expensive for $N \gg 1$ (curse of dimensionality).
- Standard quadrature slow when N large.
- Monte-Carlo better but limited to $\eta^{-1/2}$ rate.

Sparse grid quadrature

$$\mathbb{E}[\mathcal{Q}^{\varepsilon}(\mathbf{y})] \approx \sum_{k=1}^{\eta} \alpha_k \mathcal{Q}^{\varepsilon}(\mathbf{y}^{(k)}),$$

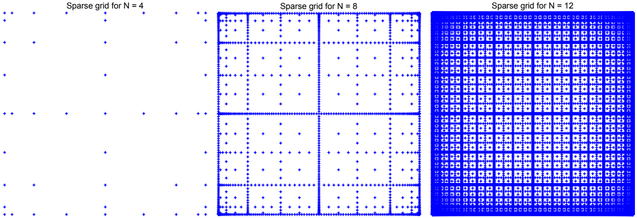
- Key point: choice of collocation point set $\{\mathbf{y}^{(k)}\}_{k=1}^{\eta} \in \Gamma$.
- Full grids expensive for $N \gg 1$ (curse of dimensionality).
- Standard quadrature slow when N large.
- Monte-Carlo better but limited to $\eta^{-1/2}$ rate.
- Sparse grids to reduce the cost.
- Sparse stochastic collocation faster if $\mathcal{Q}^{\varepsilon}$ smooth in $\mathbf{y}$.

Sparse grid quadrature



Smolyak sparse grid: nested points on Clenshaw-Curtis abscissas (extrema of Chebyshev polynomials) using total degree index set.

Sparse grid quadrature



Smolyak sparse grid: nested points on Clenshaw-Curtis abscissas (extrema of Chebyshev polynomials) using total degree index set.

- Number of collocation points grows slowly with N.
- Spectral convergence in η (number of collocation points)

$$\text{error} \leq C(p, N) M(\mathcal{Q}^\varepsilon)\, \eta^{-\frac{p}{1+\log 2N}}, \quad \forall p.$$

- Rate depends on smoothness of $\mathcal{Q}^\varepsilon$: size of $M \sim \left| \frac{d^\ell \mathcal{Q}^\varepsilon}{d\mathbf{y}^\ell} \right|$.
- Rate depends only weakly on N.

Stochastic regularity for high frequency waves

Stochastic Cauchy problem

$$u^\varepsilon_{tt}(t,\mathbf{x},\mathbf{y}) = c(\mathbf{x},\mathbf{y})^2 \Delta u^\varepsilon(t,\mathbf{x},\mathbf{y}), \qquad (t,\mathbf{x},\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Gamma,$$

$$u^\varepsilon(0,\mathbf{x},\mathbf{y}) = A_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon},$$

$$u^\varepsilon_t(0,\mathbf{x},\mathbf{y}) = \frac{1}{\varepsilon} B_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon}, \qquad t=0,\ (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \Gamma.$$

$$\mathcal{Q}^\varepsilon(\mathbf{y}) = \int_{\mathbb{R}^n} |u^\varepsilon_{GB}(T,\mathbf{x},\mathbf{y})|^2 \psi(\mathbf{x})\, d\mathbf{x}, \quad \psi \in C_c^\infty(\mathbb{R}^n).$$

- Sources of uncertainty: speed, initial position, wave phase...
- For fast convergence we need

$$\sup_{\mathbf{y}\in\Gamma} \left| \frac{d^\ell \mathcal{Q}^\varepsilon(\mathbf{y})}{d\mathbf{y}^\ell} \right| \le C_\ell, \quad \forall \ell \in \mathbb{N}^N,$$

where C_ℓ independent of the wavelength ε.

Stochastic regularity for high frequency waves

Stochastic Cauchy problem

$$u^\varepsilon_{tt}(t,\mathbf{x},\mathbf{y}) = c(\mathbf{x},\mathbf{y})^2 \Delta u^\varepsilon(t,\mathbf{x},\mathbf{y}), \qquad (t,\mathbf{x},\mathbf{y}) \in \mathbb{R}^+ \times \mathbb{R}^n \times \Gamma,$$
$$u^\varepsilon(0,\mathbf{x},\mathbf{y}) = A_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon},$$
$$u^\varepsilon_t(0,\mathbf{x},\mathbf{y}) = \frac{1}{\varepsilon} B_0(\mathbf{x},\mathbf{y}) e^{i\Phi_0(\mathbf{x},\mathbf{y})/\varepsilon}, \qquad t = 0,\ (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \Gamma.$$
$$\mathcal{Q}^\varepsilon(\mathbf{y}) = \int_{\mathbb{R}^n} |u^\varepsilon_{GB}(T,\mathbf{x},\mathbf{y})|^2 \psi(\mathbf{x})\, d\mathbf{x}, \quad \psi \in C_c^\infty(\mathbb{R}^n).$$

- Sources of uncertainty: speed, initial position, wave phase...
- For fast convergence we need

$$\sup_{\mathbf{y}\in\Gamma} \left| \frac{d^\ell \mathcal{Q}^\varepsilon(\mathbf{y})}{d\mathbf{y}^\ell} \right| \le C_\ell, \quad \forall \ell \in \mathbb{N}^N,$$

where C_ℓ independent of the wavelength ε.

Stochastic regularity for high frequency waves

In general, $u^\varepsilon(t, \mathbf{x}, \mathbf{y})$ oscillates with period $\sim \varepsilon$ both in $\mathbf{x}$ and $\mathbf{y}$.

Conjecture/Theorem

The bound

$$\sup_{\mathbf{y}\in\Gamma} \left| \frac{d^\ell \mathcal{Q}^\varepsilon(\mathbf{y})}{d\mathbf{y}^\ell} \right| \leq C_\ell, \quad \forall \ell \in \mathbb{N}^N,$$

C_ℓ independent of the wavelength ε, holds for

$$\mathcal{Q}^\varepsilon(\mathbf{y}) = \int_{\mathbb{R}^n} |u^\varepsilon_{GB}(T, \mathbf{x}, \mathbf{y})|^2 \, \psi(\mathbf{x}) \, d\mathbf{x}, \quad \psi \in C_c^\infty(\mathbb{R}^n).$$

with single family initial data.

Layout

Example 1: Caustics

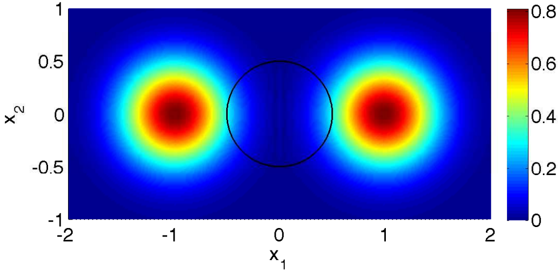


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

Example 1: Caustics

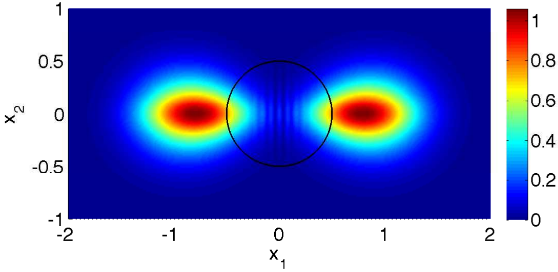


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

Example 1: Caustics

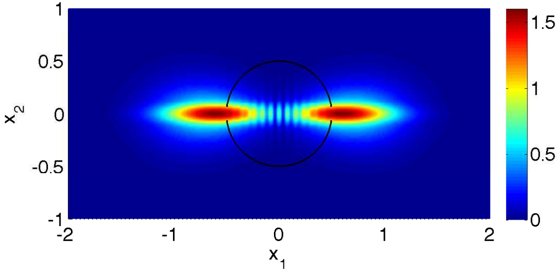


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

Example 1: Caustics

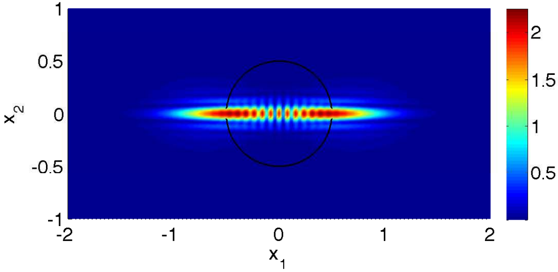


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

Example 1: Caustics

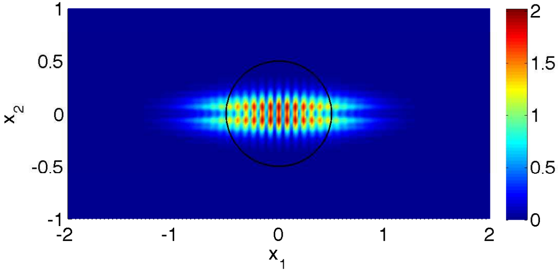


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

Example 1: Caustics

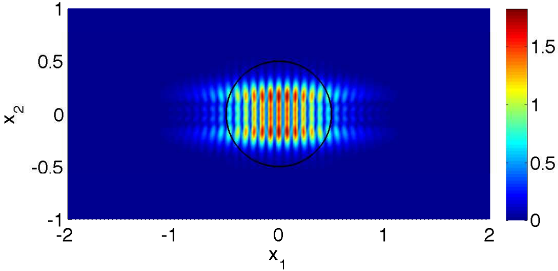


Figure: Two bumps moving towards each other (absolute value).

$$\Phi_0(\mathbf{x}) = |x_1| + x_2^2, \qquad \mathbf{x} = (x_1, x_2).$$

Caustics appear for $t \geq 0.5$. Circle indicates the support of the QoI test function.

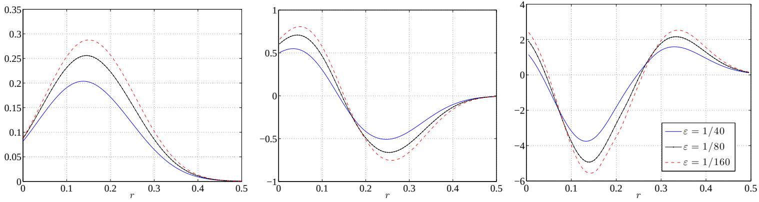
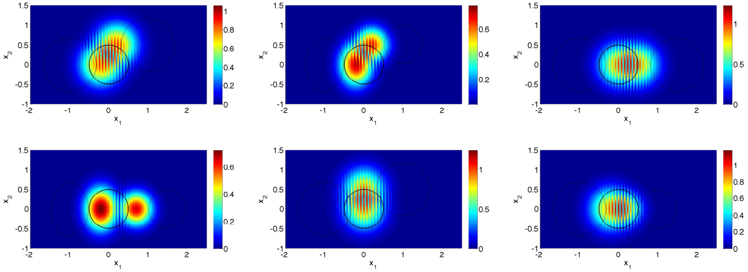


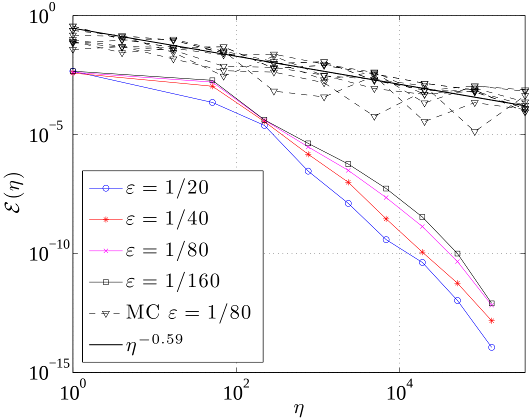
Figure : Quantity of interest $\mathcal{Q}^{\varepsilon}(\mathbf{y})$ with its first and second derivatives.

- $N = 2$ random variables – initial position (y_1) and constant speed (y_2).
- QoI along the line $\mathbf{y}(r) = (1 + r, 1 + 2r)$, for different wave lengths ε.

Example 2: Sparse grids

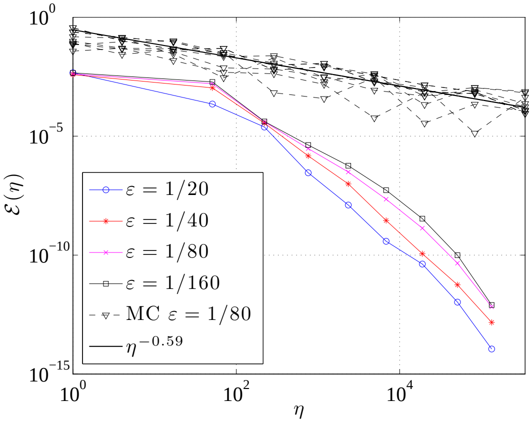


- $N = 5$ random variables (speed, initial data – pulse shape, position).
- $\Phi_0 = |x_1|$.



Relative error in the expected value of $\mathcal{Q}^{\varepsilon}$ for levels $\ell \geq 1$:

$$\mathcal{E}(\eta(\ell)) := \frac{\left|\mathbb{E}[\mathcal{S}_{\mathcal{I}(\ell_{\text{ref}})}[\mathcal{Q}^{\varepsilon}]] - \mathbb{E}[\mathcal{S}_{\mathcal{I}(\ell)}[\mathcal{Q}^{\varepsilon}]]\right|}{\left|\mathbb{E}[\mathcal{S}_{\mathcal{I}(\ell_{\text{ref}})}[\mathcal{Q}^{\varepsilon}]]\right|}.$$



- Fast spectral convergence compared to Monte-Carlo.
- As ε decreases, error converges $\Rightarrow$ uniform bounds.

THANK YOU FOR YOUR ATTENTION