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# Multilevel Subset Simulation for rare event probabilities

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# Introduction: **Rare Event Probabilities**

### Definition: Rare event

- $G : \Omega \to \mathbb{R}$ – The limit state functiona, stochastic variable
- The rare event probability $P$ is:

$$P = \Pr(G \leq 0)$$

where $P \ll 1$ is unknown but **small**.

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# Introduction: **Problem formulation**

### Model problem

- $\mathcal{M}$ – model
- $\Omega$ – sample space

We assume that there exists a unique solution $u$ given any $\omega \in \Omega$:

$$\mathcal{M}(\omega, u(\omega)) = 0.$$

- $G(u) : V \to \mathbb{R}$ – Limit state function (functional) of the solution $u$
- The solution $u$ is uniquely determined by the data, $G(\omega) := G(u(\omega))$

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# Introduction: **Numerical errors**

### Assumption: Numerical error for samples

- For $\omega \in \Omega$ the numerical approximation $G_\ell(\omega)$ of $G(\omega)$ satisfies

$$|G(\omega) - G_\ell(\omega)| \le \epsilon_\ell,$$

  for $\epsilon_\ell > 0$

- The work $W$ for computing $G_\ell(\omega_i)$ satisfy

$$W(G(\omega_i)) = \mathcal{O}(\epsilon_\ell^{-q}),$$

  where $q > 0$ are independent of $\omega_i$

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# Introduction: **Monte Carlo for rare events estimation**

- Large stochastic dimension $\Rightarrow$ Monte Carlo based method
- Standard Monte Carlo (MC) methods need a large number of samples:

$$\frac{\sqrt{V[\widehat{P}_{MC}]}}{E[\widehat{P}_{MC}]} \leq \epsilon \quad \Longrightarrow \quad N \geq \frac{(1-P)}{P\epsilon^2} \sim P^{-1}$$

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Idea
Work estimate

# Subset simulation: **Idea**

- Subset simulation – A variance reduction techniques (Au and Beck, 2001)
- Idea: Uses Bayes theorem

$$\Pr(B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(A|B)}$$

if $B \subset A \Rightarrow \Pr(A|B) = 1$ and we have

$$\Pr(B) = \Pr(B|A)\Pr(A).$$

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**Idea**
Work estimate

- Define $\widetilde{F}_\ell = G_L(\omega) \leq c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- We have that $\widetilde{F}_{\ell+1} \subset \widetilde{F}_\ell$ are subsets
- Applying Bayes theorem on the sequence of subsets, $\widetilde{F}_L \subset \widetilde{F}_{L-1} \subset \cdots \subset \widetilde{F}_0$, yields

$$\Pr(G_L(\omega) \leq 0) = \Pr(\widetilde{F}_L) = \Pr(\widetilde{F}_0) \prod_{\ell=1}^{L} \Pr(\widetilde{F}_\ell | \widetilde{F}_{\ell-1})$$

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# Subset simulation: **Work estimate**

- Choosing $\Pr(\widetilde{F}_\ell | \widetilde{F}_{\ell-1}) \sim 0.1$ we get
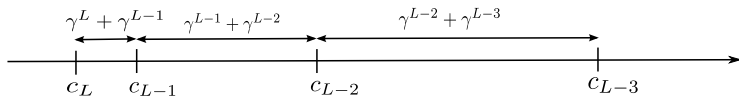
$$P \sim 0.1^L$$

  and $L \sim \log(P^{-1})$.

- Number of samples for each failure event $\Pr(\widetilde{F}_\ell | \widetilde{F}_{\ell-1})$ is independent of $P$

- Hence, total number of samples

$$N \sim \log(P^{-1})$$

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**Subset property**
Estimate numerical bias
Algorithm

# Multilevel subset simulation: **Subset property**

- Multilevel approach has been proven successful in other application, e.g., multigrid and multilevel Monte Carlo
- Define $F_\ell = G_\ell(\omega) \le c_\ell$, where $c_0 > c_1 > \cdots > c_L = 0$
- Different model resolution: $\Pr(F_\ell \mid F_{\ell+1}) \ne 1$
- Fix: $\Pr(F_\ell \mid F_{\ell+1}) = 1$ if $|c_\ell - c_{\ell+1}| > \gamma^\ell + \gamma^{\ell+1}$

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# Multilevel subset simulation: **Estimate numerical bias**

- Recall: $\Pr(F_L) = \Pr(F_1) \prod_{\ell=2}^{L} \Pr(F_\ell | F_{\ell-1})$
- Lets define: $F_\infty = G_\infty < c_0$, then

$$P = \Pr(F_\infty) = \Pr(F_1) \Pr(F_\infty | F_{L-1}) \prod_{\ell=2}^{L-1} \Pr(F_\ell | F_{\ell-1})$$

  since $\Pr(F_{L-1} | F_\infty) = 1$

- The numerical bias can be estimated

$$|P - \Pr(F_L)| = \Pr(F_1) \prod_{\ell=2}^{L-1} \Pr(F_\ell | F_{\ell-1}) |Pr(F_\infty | F_{L-1}) - Pr(F_L | F_{L-1})|$$

$$\approx \frac{\widehat{P}}{\widehat{P}_{F_L | F_{L-1}}} \left| \widehat{P}_{F_{L+1} | F_{L-1}} - \widehat{P}_{F_L | F_{L-1}} \right|$$

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# Multilevel subset simulation: **Algorithm**

**Input:** $L$ (number of layers) and $\mathrm{TOL}$ (desired tolerance)
Determine intermediate subsets $F_\ell$
Set $\epsilon = \mathrm{TOL}/(2L^{1/2})$
**for** $\ell = 1, \ldots, L$ **do**
    Estimate $\widehat{P}_\ell \approx \Pr(F_\ell \mid F_{\ell-1})$ such that $e(\widehat{P}_\ell) \leq \epsilon$
**end for**
Set the subset estimator to $\widehat{P} = \prod_{i=1}^{L} \widehat{P}_i$
Estimate error $e(\widehat{P})$

- Estimate $\widehat{P}_1$ using i.i.d samples or Standard subset simulation
- Estimate $\widehat{P}_\ell$ for $\ell > 1$ using conditional samples
  (Metropolis-Hasting)

# Numerical experiment: **Model problem**
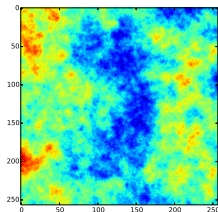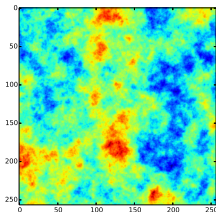
- One phase flow model on a unit square:

$$-\nabla \cdot A(\omega)\nabla u = 0,$$

with Boundary condition No-flow up and down, $u = 1$ on $\Gamma$ left boundary and $u = 0$ right

- Limit state function

$$G(\omega) = 3.1 - \int_\Gamma \nu \cdot A(\omega)\nabla u \, \mathrm{d}x$$

- Permeability – $A(\omega)$ is a Gaussian random field with exponential covariance with correlation length $\lambda = 0.1$

# Numerical experiment: **Setup**

**Settings:**

- Set $L = 4$, $\gamma = 1/2$, $\mathrm{TOL} = 0.5$, and 5 Markov chains

**Estimators:**

- $\mathrm{SS}(\mathrm{TOL})$ (Standard subset simulation)
- $\mathrm{ML}_1(L, \mathrm{TOL})$ (Multilevel with i.i.d samples on level 1)
- $\mathrm{ML}_2(L, \mathrm{TOL})$ (Multilevel with subset simulation on level 1)

# Numerical experiment: **$ML_1$ estimator**

- Mean number of samples and mean cost on each level from 10 runs of the multilevel subset simulation estimator $ML_1$

| $\ell$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\widehat{P}_\ell$ | 0.00028 | 0.064 | 0.18 | 0.38 |
| $N_{\ell-1}$ | 0 | 486 | 297 | 260 |
| $N_\ell$ | 41857 | 87 | 37 | 24 |
| Cost | 41857 | 834 | 1780 | 5696 |

# Numerical experiment: **$ML_2$ estimator**

- Mean number of samples and mean cost on each level from 10 runs of the multilevel subset simulation estimator $ML_2$

| $\ell$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\widehat{P}_\ell$ | 0.00035 | 0.13 | 0.31 | 0.31 |
| $N_{\ell-1}$ | 0 | 193 | 150 | 159 |
| $N_\ell$ | 912 | 64 | 31 | 29 |
| Cost | 912 | 448 | 1096 | 4400 |

# Numerical experiment: **Comparison**

- Repeat all estimators 10 times to get statistial data
- Standard Monte Carlo cost is estimated

|      | $ML_2$ | $ML_1$ | SS | MC   |
|------|--------|--------|----|------|
| Cost | 1      | 7.3    | 13 | 3011 |

# Numerical experiment: **Propertie of the $ML_2$ estimator**

- Repeat estimator 100 times to get statistial data
- The estimate mean rare event probability and relative variance are:

$$\mathbb{E}\left[\widehat{P}^{\mathrm{ML}_2}\right] \approx 2.29 \cdot 10^{-6} \quad \text{and} \quad \frac{\left(\mathbb{V}\left[\widehat{P}^{\mathrm{ML}_2}\right]\right)^{1/2}}{\mathbb{E}\left[\widehat{P}^{\mathrm{ML}_2}\right]} \approx 0.71 \leq 1$$

- The estimated relative variance of the estimator is less than the tolerance $\mathrm{TOL} = 1$.