UPPSALA
UNIVERSITET

Stabilized Nitsche Cut-Elements for Wave Propagation

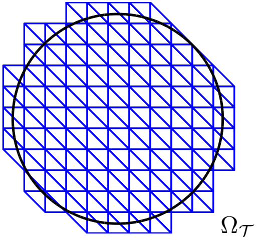
Simon Sticko, Gunilla Kreiss

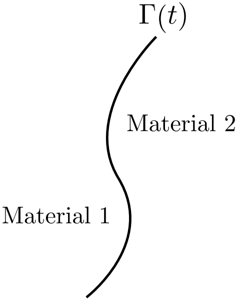
26 August 2015

Introduction

UPPSALA UNIVERSITET

- Cut elements - Immersed method
- Domain Ω
- Uniform background mesh
- Application: Interface moving with time
  - Distorted mesh
  - Remeshing expensive





Main Goal

UPPSALA
UNIVERSITET

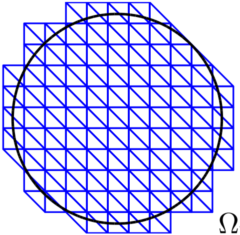
- Immersed method for Wave Propagation
- Focus: Scalar wave eq
- Future: Elastic wave eq
- High order more effective

Problem

UPPSALA UNIVERSITET

- Wave equation

$$\begin{cases} \ddot{u} & = \nabla^2 u + f(\mathbf{x}, t) \quad \mathbf{x} \in \Omega \\ u\big|_{\Gamma_D} & = g_D(\mathbf{x}, t) \\ \frac{\partial u}{\partial n}\Big|_{\Gamma_N} & = g_N(\mathbf{x}, t). \end{cases}$$



- FEM, Based on Burman and Hansbo [2012]
- Solve for dofs of $\Omega_{\mathcal{T}}$
- Weak boundary conditions by Nitsche

$$(\ddot{u}, v)_\Omega + a(u, v) = L(v) \;\; \forall v \in V,$$

$$a(u,v) = (\nabla u, \nabla v)_\Omega - \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_D} \overbrace{- \left\langle u, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_D} + \beta_D \langle u, v \rangle_{\Gamma_D} + \beta_N \left\langle \frac{\partial u}{\partial n}, \frac{\partial v}{\partial n} \right\rangle_{\Gamma_N}}^{\text{Nitsche penalty terms}}$$

$$L(v) = (f, v)_\Omega + \overbrace{\left\langle g_D, \beta_D v - \frac{\partial v}{\partial n} \right\rangle_{\Gamma_D} + \left\langle g_N, v + \beta_N \frac{\partial v}{\partial n} \right\rangle_{\Gamma_N}}^{\text{Nitsche penalty terms}}.$$

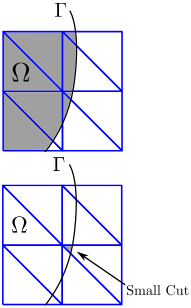
Ill-conditioning

- Want to calculate quantities of type $(\phi_i, \phi_j)_\Omega$, $(\nabla\phi_i, \nabla\phi_j)_\Omega$, Ill-conditioned
- Stabilization, j, Burman and Hansbo [2012], Massing et al. [2014]

Weak fom: $(\ddot{u}, v)_\Omega + a(u, v) = L(v) \ \forall v \in V,$

FEM: $M(\ddot{u}_h, v_h) + A(u_h, v_h) = L(v_h) \ \forall v_h \in V_h$

$$M(\ddot{u}_h, v_h) := (\ddot{u}_h, v_h)_\Omega + \gamma_M h^2 j(\ddot{u}_h, v_h)$$
$$A(u_h, v_h) := a(u_h, v_h) + \gamma_A j(u_h, v_h)$$



$$\|v_h\|^2_{\Omega_\mathcal{T}} \leq C_M M(v_h, v_h) \leq C_M \|v_h\|^2_{\Omega_\mathcal{T}}$$

$A(u_h, v_h)$ − coercive

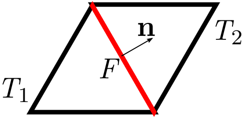
Stabilization

UPPSALA UNIVERSITET

- Estimate, by Taylor expansion

$$\|v\|_{T_1}^2 \leq C\left(\|v\|_{T_2}^2 + \sum_{q=1}^{p} \frac{1}{q!^2}\frac{h^{2q+1}}{2q+1}\left\langle[\partial_{\mathbf{n}}^q v], [\partial_{\mathbf{n}}^q v]\right\rangle_F\right)$$

$$\partial_{\mathbf{n}}^q v = \sum_{|\alpha|=q} \frac{q!}{\alpha!} D^\alpha v \mathbf{n}^\alpha$$



$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$$

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$$

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x^{\alpha_d}}$$

$$\mathbf{n}^\alpha = n_1^{\alpha_1} n_2^{\alpha_2} ... n_d^{\alpha_d}$$

$$\alpha! = \alpha_1! \alpha_2! ... \alpha_n!$$

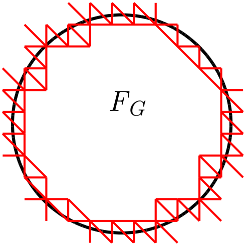
Stabilization

UPPSALA UNIVERSITET

- Stabilization term j on faces F_G

$$j(u,v) = \sum_{F \in F_G} \sum_{q=1}^{p} h^{2q-1} \left\langle [\partial_{\mathbf{n}}^q u] \, , [\partial_{\mathbf{n}}^q v] \right\rangle_F$$

$$\text{for } p = 1: \; j(u,v) = \sum_{F \in F_G} \langle h[n \cdot \nabla u_h], [n \cdot \nabla v_h] \rangle_F$$



Symmetry

UPPSALA UNIVERSITET

$$M(\ddot{u}_h, v_h) + A(u_h, v_h) = L(v_h) \ \forall v_h \in V_h$$

- Discretizes to

$$\mathcal{M}\ddot{\xi} + \mathcal{A}\xi = \mathcal{L}(t)$$

- $\mathcal{M}$ sym pos def $\Rightarrow \mathcal{M}^{\frac{1}{2}}$ well defined

$$\ddot{\eta} + \mathcal{B}\eta = \mathcal{F}(t)$$

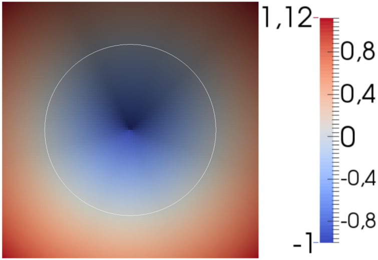
where

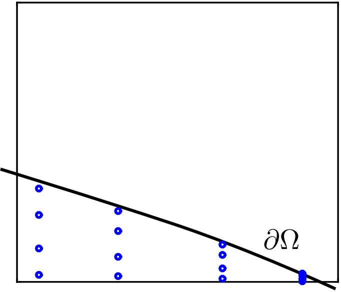
$$\eta = \mathcal{M}^{\frac{1}{2}}\xi, \ \ \mathcal{B} = \mathcal{M}^{-\frac{1}{2}}\mathcal{A}\mathcal{M}^{-\frac{1}{2}}$$

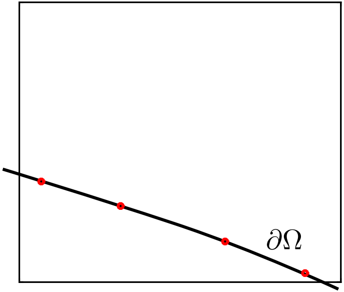
- Real wave-numbers!

Quadrature

- Levelset
  - $\partial\Omega$ def by zero contour
- Quadrature
  - Use algorithm from Saye [2015]
  - Hyperrectangels
  - Build Quadrature from levelset
  - Implicit function theorem
  - 1D-rootfinding



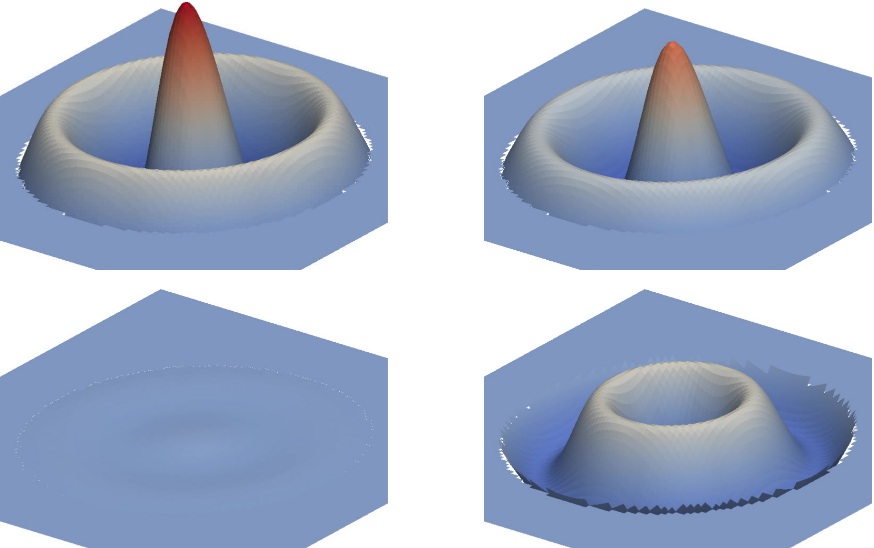




Membrane

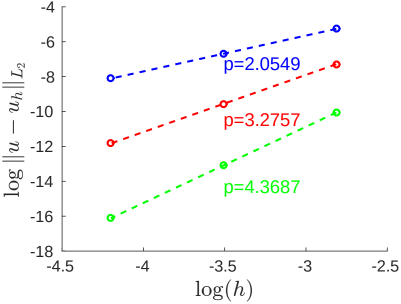
UPPSALA UNIVERSITET

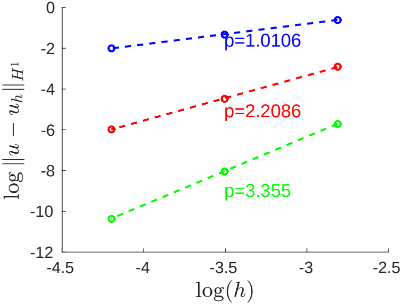
- Circular domain
- $u|_{\partial\Omega} = 0$
- $u|_{t=0} = J_0(\alpha_n \frac{\|\mathbf{x}\|}{R})$
- $\dot{u}|_{t=0} = 0$

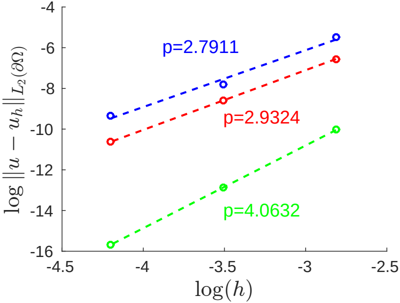


Convergence Q1-Q3 elements

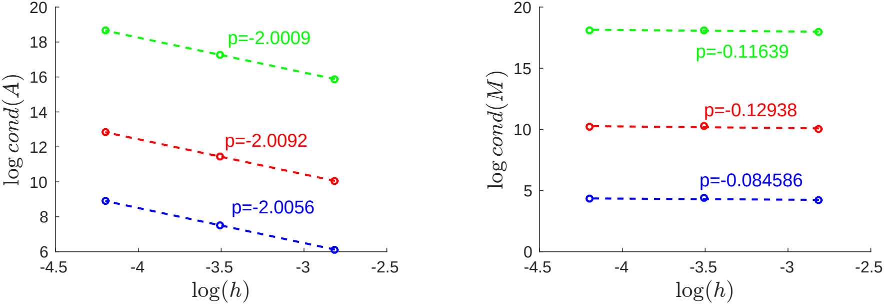
UPPSALA
UNIVERSITET







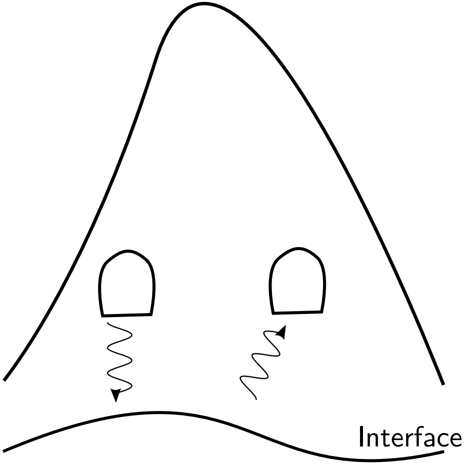
Condition Numbers Q1-Q3 elements



Future Application

UPPSALA UNIVERSITET

- Elastic wave equation
- Unknown interface geometry
- Find shape by optimization



Summary

UPPSALA UNIVERSITET

- Cut elements for wave propagation
- Weak boundary condition by Nitsche
- Stabilizing terms
- Tested up to 4th order convergence
- Plans for elastic wave equation

UPPSALA
UNIVERSITET

Thank you!

Erik Burman and Peter Hansbo. Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method. *Applied Numerical Mathematics*, 62(4):328–341, 2012. ISSN 0168-9274. doi: 10.1016/j.apnum.2011.01.008. URL http://dx.doi.org/10.1016/j.apnum.2011.01.008http://www.sciencedirect.com/science/article/pii/S0168927411000298.

André Massing, Mats G. Larson, Anders Logg, and Marie E. Rognes. A Stabilized Nitsche Fictitious Domain Method for the Stokes Problem. *Journal of Scientific Computing*, pages 604–628, 2014. ISSN 08857474. doi: 10.1007/s10915-014-9838-9.

R. I. Saye. High-Order Quadrature Methods for Implicitly Defined Surfaces and Volumes in Hyperrectangles. *SIAM Journal on Scientific Computing*, 37(2):A993–A1019, April 2015. ISSN 1064-8275. doi: 10.1137/140966290. URL http://epubs.siam.org/doi/ref/10.1137/140966290.