Coupled Sylvester-type Matrix Equations and Block Diagonalization

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BIT Circus, Umeå
August 26, 2015

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Generalized Roth's Theorem I (D. & Kågström, 2015)

System of the Sylvester matrix equations

$$A_i X_k - X_j B_i = C_i, \quad i = 1, \dots, n$$

has a solution $X_1, \dots, X_m$.

$$\Updownarrow$$

There exist non-singular matrices $P_1, \dots, P_m$ such that

$$P_j^{-1} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} P_k = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, n.$$

Roth's Theorem I, 1952

The Sylvester-type matrix equation

$$AX - XB = C, \quad A \text{ is } m \times m,\ B \text{ is } n \times n,\ \text{and}\ C \text{ is } m \times n,$$

has a solution.

$$\Updownarrow$$

There exists a non-singular matrix P such that

$$P^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Roth's Theorem II, 1952

The Sylvester-type matrix equation

$$AX_1 - X_2B = C,$$

has a solution.

$$\Updownarrow$$

There exist non-singular matrices P_1 and P_2 such that

$$P_2^{-1}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Roth's Theorem III, 1994 (2 papers), 1996

The system of Sylvester-type matrix equations

$$A_1 X_1 - X_2 B_1 = C_1,$$
$$A_2 X_1 - X_2 B_2 = C_2$$

has a solution.

$$\Updownarrow$$

There exist non-singular matrices P_1 and P_2 such that

$$P_2^{-1} \begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$
$$P_2^{-1} \begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_1 = \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Change of basis in a vector space

We have $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} x = y$.

We change the basis in **V** : $\quad Px' = x$ and $Py' = y$.

We obtain $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} Px' = Py'$.

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \qquad\qquad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

‘old basis’ ‘new basis’

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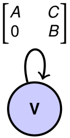
$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \qquad\qquad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

‘old basis’ ‘new basis’

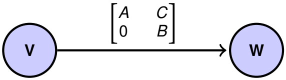
$$P^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Graphs associated with Roth's Theorems

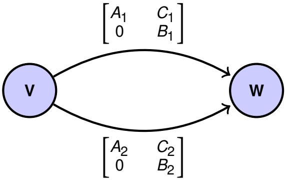
$$AX - XB = C \quad \overset{'52,'77}{\Longleftrightarrow} \quad P^{-1}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \Leftrightarrow$$



$$AX_1 - X_2 B = C \quad \overset{'52,'77}{\Longleftrightarrow} \quad P_2^{-1}\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \quad \Leftrightarrow$$

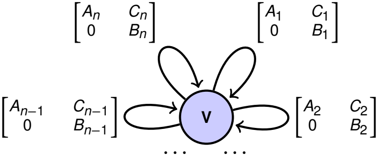


$$\begin{aligned} A_1 X_1 - X_2 B_1 &= C_1 \\ A_2 X_1 - X_2 B_2 &= C_2 \end{aligned} \quad \overset{'94,'96}{\Longleftrightarrow} \quad \begin{aligned} P_2^{-1}\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 &= \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} \\ P_2^{-1}\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_1 &= \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix} \end{aligned} \quad \Leftrightarrow$$

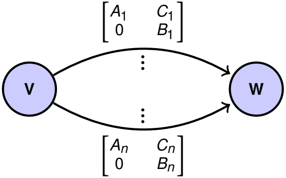


Graphs associated with Roth's Theorems

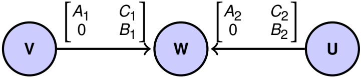
$$A_i X - X B_i = C_i,\quad i = 1,\dots,n \quad \overset{'85,'12}{\Longleftrightarrow} \quad P^{-1}\begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} P = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix},\quad i = 1,\dots,n \quad \Leftrightarrow$$



$$A_i X_1 - X_2 B_i = C_i,\ '85,'94,'12 \quad i = 1,\dots,n \quad \Longleftrightarrow \quad P_2^{-1}\begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} P_1 = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix},\quad i = 1,\dots,n \quad \Leftrightarrow$$

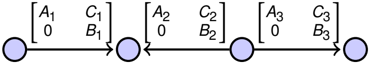


$$\begin{aligned} A_1 X_1 - X_2 B_1 &= C_1 \\ A_2 X_3 - X_2 B_2 &= C_2 \end{aligned} \quad \overset{'12}{\Longleftrightarrow} \quad \begin{aligned} P_2^{-1}\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 &= \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} \\ P_2^{-1}\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_3 &= \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix} \end{aligned} \quad \Leftrightarrow$$

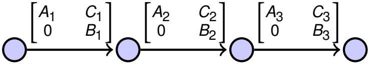


Graphs associated with Roth's Theorems, 2014

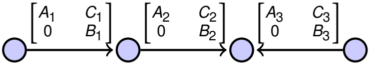
$$\begin{aligned} A_1 X_1 - X_2 B_1 &= C_1 \\ A_2 X_3 - X_2 B_2 &= C_2 \\ A_3 X_3 - X_4 B_3 &= C_3 \end{aligned} \iff$$



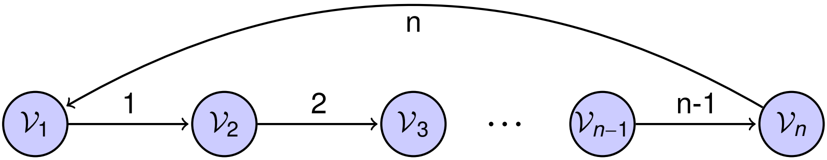
$$\begin{aligned} A_1 X_1 - X_2 B_1 &= C_1 \\ A_2 X_2 - X_3 B_2 &= C_2 \\ A_3 X_3 - X_4 B_3 &= C_3 \end{aligned} \iff$$



$$\begin{aligned} A_1 X_1 - X_2 B_1 &= C_1 \\ A_2 X_2 - X_3 B_2 &= C_2 \\ A_3 X_4 - X_3 B_3 &= C_3 \end{aligned} \iff$$



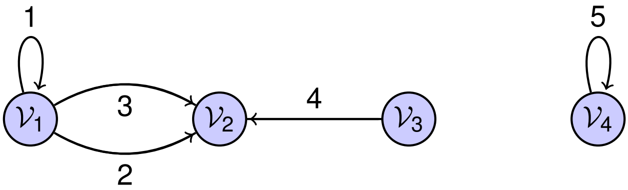
Cyclic graphs (Periodic eigenvalue problem)



System of matrix equations is consistent $\Leftrightarrow$ Relation on matrices

$$\begin{array}{lll} 1: & A_1 X_1 - X_2 B_1 = C_1, & P_2^{-1}\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \\ 2: & A_2 X_2 - X_3 B_2 = C_2, & P_3^{-1}\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_2 = \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}, \\ 3: & A_3 X_3 - X_4 B_3 = C_3, \quad \Leftrightarrow & P_4^{-1}\begin{bmatrix} A_3 & C_3 \\ 0 & B_3 \end{bmatrix} P_3 = \begin{bmatrix} A_3 & 0 \\ 0 & B_3 \end{bmatrix}, \\ & \cdots & \cdots \\ n: & A_n X_n - X_1 B_n = C_n, & P_1^{-1}\begin{bmatrix} A_n & C_n \\ 0 & B_n \end{bmatrix} P_n = \begin{bmatrix} A_n & 0 \\ 0 & B_n \end{bmatrix}. \end{array}$$

Graphs associated with Generalized Roth's Theorem I



System of matrix equations is consistent ⇔ Relation on matrices

$$\begin{aligned}
1:&\quad A_1X_1 - X_1B_1 = C_1, & & P_1^{-1}\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},\\
2:&\quad A_2X_1 - X_2B_2 = C_2, & & P_2^{-1}\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_1 = \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix},\\
3:&\quad A_3X_1 - X_2B_3 = C_3, & \Leftrightarrow & P_2^{-1}\begin{bmatrix} A_3 & C_3 \\ 0 & B_3 \end{bmatrix} P_1 = \begin{bmatrix} A_3 & 0 \\ 0 & B_3 \end{bmatrix},\\
4:&\quad A_4X_3 - X_2B_4 = C_4, & & P_2^{-1}\begin{bmatrix} A_4 & C_4 \\ 0 & B_4 \end{bmatrix} P_3 = \begin{bmatrix} A_4 & 0 \\ 0 & B_4 \end{bmatrix},\\
5:&\quad A_5X_4 - X_4B_5 = C_5, & & P_4^{-1}\begin{bmatrix} A_5 & C_5 \\ 0 & B_5 \end{bmatrix} P_4 = \begin{bmatrix} A_5 & 0 \\ 0 & B_5 \end{bmatrix}.
\end{aligned}$$

Generalized Roth's Theorem II (D. & Kågström, 2015)

System of the Sylvester and $\star$-Sylvester matrix equations

$$A_i X_k - X_j B_i = C_i, \quad i = 1, \ldots, n_1$$
$$F_{i'} X_{k'} - X_{j'}^{\star} G_{i'} = H_{i'}, \quad i' = 1, \ldots, n_2$$

has a solution $X_1, \ldots, X_m$.

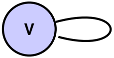
$$\Updownarrow$$

There exist non-singular matrices $P_1, \ldots, P_m$ such that

$$P_j^{-1} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} P_k = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad i = 1, \ldots, n_1,$$
$$P_{j'}^{\star} \begin{bmatrix} 0 & G_{i'} \\ F_{i'} & H_{i'} \end{bmatrix} P_{k'} = \begin{bmatrix} 0 & G_{i'} \\ F_{i'} & 0 \end{bmatrix}, \quad i = 1, \ldots, n_2.$$

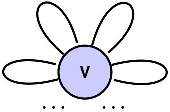
Graphs associated with Roth's Theorems

$$FX - X^\star G = H \quad \overset{'94,'11}{\Longleftrightarrow} \quad P^\star \begin{bmatrix} 0 & G \\ F & H \end{bmatrix} P = \begin{bmatrix} 0 & G \\ F & 0 \end{bmatrix} \quad \Leftrightarrow$$



$$\begin{bmatrix} 0 & G \\ F & H \end{bmatrix}$$

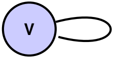
$$F_i X - X^\star G_i = H_i, \quad i = 1, \ldots, n \quad \overset{'14}{\Longleftrightarrow} \quad P^\star \begin{bmatrix} 0 & G_i \\ F_i & H_i \end{bmatrix} P = \begin{bmatrix} 0 & G_i \\ F_i & 0 \end{bmatrix}, \quad i = 1, \ldots, n \quad \Leftrightarrow$$



$$\begin{bmatrix} 0 & G_n \\ F_n & H_n \end{bmatrix} \quad \begin{bmatrix} 0 & G_1 \\ F_1 & H_1 \end{bmatrix} \quad \begin{bmatrix} 0 & G_{n-1} \\ F_{n-1} & H_{n-1} \end{bmatrix} \quad \begin{bmatrix} 0 & G_2 \\ F_2 & H_2 \end{bmatrix}$$

Graphs associated with Roth's Theorems

$$FX - X^\star G = H \quad \overset{'94,'11}{\Longleftrightarrow} \quad P^\star \begin{bmatrix} 0 & G \\ F & H \end{bmatrix} P = \begin{bmatrix} 0 & G \\ F & 0 \end{bmatrix} \quad \Leftrightarrow$$

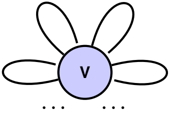


$$\begin{bmatrix} 0 & G \\ F & H \end{bmatrix}$$

$$F_i X - X^\star G_i = H_i, \; i = 1, \dots, n \quad \overset{'14}{\Longleftrightarrow} \quad P^\star \begin{bmatrix} 0 & G_i \\ F_i & H_i \end{bmatrix} P = \begin{bmatrix} 0 & G_i \\ F_i & 0 \end{bmatrix}, \; i = 1, \dots, n \quad \Leftrightarrow$$

$$\begin{bmatrix} 0 & G_n \\ F_n & H_n \end{bmatrix} \quad \begin{bmatrix} 0 & G_1 \\ F_1 & H_1 \end{bmatrix}$$

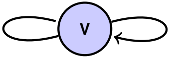
$$\begin{bmatrix} 0 & G_{n-1} \\ F_{n-1} & H_{n-1} \end{bmatrix}$$



$$\begin{bmatrix} 0 & G_2 \\ F_2 & H_2 \end{bmatrix}$$

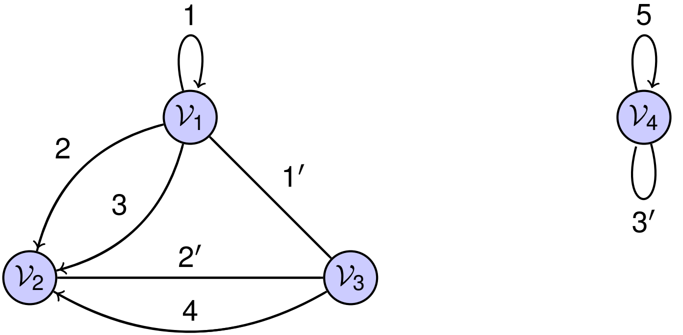
$$\begin{array}{c} AX - XB = C, \\ X - X^\star = 0 \end{array} \quad \overset{'94}{\Longleftrightarrow} \quad \begin{array}{c} P^{-1} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} P = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \\ P^\star \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} P = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \end{array} \quad \Leftrightarrow$$

$$\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$



$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

Graphs associated with Generalized Roth's Theorem II



System of matrix equations is consistent ⇔ Relation on matrices

Particular case of Generalized Roth's Theorem II

$$
\begin{aligned}
1: &\quad A_1 X_1 - X_1 B_1 = C_1, & & P_1^{-1}\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} P_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \\
2: &\quad A_2 X_1 - X_2 B_2 = C_2, & & P_2^{-1}\begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} P_1 = \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}, \\
3: &\quad A_3 X_1 - X_2 B_3 = C_3, & & P_2^{-1}\begin{bmatrix} A_3 & C_3 \\ 0 & B_3 \end{bmatrix} P_1 = \begin{bmatrix} A_3 & 0 \\ 0 & B_3 \end{bmatrix}, \\
4: &\quad A_4 X_3 - X_2 B_4 = C_4, & & P_2^{-1}\begin{bmatrix} A_4 & C_4 \\ 0 & B_4 \end{bmatrix} P_3 = \begin{bmatrix} A_4 & 0 \\ 0 & B_4 \end{bmatrix}, \\
5: &\quad A_5 X_4 - X_4 B_5 = C_5, & \Leftrightarrow & P_4^{-1}\begin{bmatrix} A_5 & C_5 \\ 0 & B_5 \end{bmatrix} P_4 = \begin{bmatrix} A_5 & 0 \\ 0 & B_5 \end{bmatrix}, \\
1': &\quad F_1 X_3 + X_1^\star G_1 = H_1, & & P_1^\star\begin{bmatrix} 0 & G_1 \\ F_1 & H_1 \end{bmatrix} P_3 = \begin{bmatrix} 0 & G_1 \\ F_1 & 0 \end{bmatrix}, \\
2': &\quad F_2 X_2 + X_3^\star G_2 = H_2, & & P_3^\star\begin{bmatrix} 0 & G_2 \\ F_2 & H_2 \end{bmatrix} P_2 = \begin{bmatrix} 0 & G_2 \\ F_2 & 0 \end{bmatrix}, \\
3': &\quad F_3 X_4 + X_4^\star G_3 = H_3, & & P_4^\star\begin{bmatrix} 0 & G_3 \\ F_3 & H_3 \end{bmatrix} P_4 = \begin{bmatrix} 0 & G_3 \\ F_3 & 0 \end{bmatrix}.
\end{aligned}
$$

Systems of Stein and $\star$-Stein matrix equations

$$A_i X_k K_i - L_i X_j B_i = C_i, \quad i = 1, \ldots, n_1$$
$$F_{i'} X_{k'} M_{i'} - N_{i'} X_{j'}^{\star} G_{i'} = H_{i'}, \quad i' = 1, \ldots, n_2$$

with unknown matrices $X_1, \ldots, X_m$.

A. Dmytryshyn and B. Kågström, Coupled Sylvester-type matrix equations and block diagonalization, SIAM J. Matrix Anal. Appl., 36(2) (2015) 580–593.

Thank you!