Schur complement matrix and its (elementwise) approximation: A spectral analysis based on GLT sequences

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Outline of the talk

- Target problem from Geophysics
- Block preconditioner and Schur complement
- Toeplitz theory
- Results
- Conclusion

Target Problem

Glacial Isostatic Adjustment

Simplified elastic model

$$-\nabla \cdot \left(2\mu_E \varepsilon\right) - \nabla\left(u \cdot b\right) + \left(\nabla \cdot u\right)c - \mu_E \nabla p = f,$$
$$\mu_E \nabla \cdot u - \frac{\mu_E^2}{\lambda_E} p = 0.$$

The model is discretized using:

- Modified Taylor-Hood Q1Q1. (a stable finite element pair of spaces)
- quadrilateral mesh

$$\mathcal{A} = \begin{bmatrix} K & B^T \\ B & -M \end{bmatrix}$$

- The pivot block K could be nonsymmetric.
- M is a scaled mass matrix.

Preconditioning

Schur complement

Iterative solution of large scale linear systems:

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \text{ Seek } \mathcal{D} : \mathcal{D}^{-1}\mathcal{A}\mathbf{x} = \mathcal{D}^{-1}\mathbf{b}$$

Use a block factorization of $\mathcal{A}$

$$\mathcal{A} = \begin{bmatrix} K & B^T \\ B & -M \end{bmatrix} = \begin{bmatrix} K & 0 \\ B & -S_{\mathcal{A}} \end{bmatrix} \begin{bmatrix} I_1 & K^{-1}B^T \\ 0 & I_2 \end{bmatrix},$$
$$S_{\mathcal{A}} = M + BK^{-1}B^T.$$

Many possibilities to construct preconditioners. One possibility is

$$\mathcal{D} = \begin{bmatrix} K & 0 \\ B & -S_{\mathcal{A}} \end{bmatrix}$$

- ✓ $\mathcal{D}$ is the ideal preconditioner
- ✓ the GMRES solver, in certain cases, converges in maximum two iterations when preconditioned with $\mathcal{D}$.
- ✗ $\mathcal{D}$ is expensive to construct and computationally inefficient. Involves an inverse of K.

The action of the preconditioner $\mathcal{D}$

Matrix $\mathcal{A} = \begin{bmatrix} K & B^T \\ B & -M \end{bmatrix}$, preconditioned by $\mathcal{D} = \begin{bmatrix} [K] & 0 \\ B & -[\widetilde{S}_{\mathcal{A}}] \end{bmatrix}$

- $\widetilde{S}_{\mathcal{A}}$ approximates $S_{\mathcal{A}} = M + BF^{-1}B^T$
- $[\cdot]$ means a (preconditioned, very efficient) inner solver.

From theory: very few (outer) iterations if

- K is solved accurately.
- $\widetilde{S}_{\mathcal{A}}$ is a good approximation for $S_{\mathcal{A}}$.

The Schur complement approximation

Compute $\widetilde{S}_{\mathcal{A}}$ using the *element-wise* approach

$$\mathcal{A} = \sum_{i=1}^{N} W_i^T \mathcal{A}_i^e W_i, \quad \text{where} \quad \mathcal{A}_i^e = \begin{bmatrix} K_i^e & (B_i^e)^T \\ B_i^e & -M_i^e \end{bmatrix},$$

W_i local-to-global mapping of the degrees of freedom

N number of cells in the discretization.

$$S_i^e = M_i^e + B_i^e \left(K_i^e + h_i^2 I \right)^{-1} (B_i^e)^T,$$

$$\widetilde{S}_{\mathcal{A}} = \sum_{i=1}^{N} (W_{i:2})^T S_i^e W_{i:2}$$

- h - characteristic size of the spatial mesh.

Analysis of the element-wise Schur

✓ $\mathcal{A}$ - SPD; $\rightarrow 0 < \alpha S_{\mathcal{A}} \leq \widetilde{S}_{\mathcal{A}} \leq S_{\mathcal{A}}$

Bounds have been derived earlier (J. Kraus, O. Axelsson, R. Blaheta, M. Neytcheva).

✓ $\mathcal{A}$ - symmetric indefinit; $\rightarrow 0 < S_{\mathcal{A}} \leq \widetilde{S}_{\mathcal{A}} \leq \beta S_{\mathcal{A}}$.

Shown to work. Not proven. Erik Bängtsson, Maya Neytcheva (2006-2007)

A. Dorostkar, M. Neytcheva and B. Lund. Numerical and computational aspects of some block-preconditioners for saddle point systems. Journal of Parallel Computing, 2015, In press.

✗ $\mathcal{A}$ - non-symmetric (new tools needed).

Toeplitz

A short introduction

Toeplitz

A Toeplitz matrix T_n generated by $f(\theta)$ is a square matrix that has constant blocks or elements along each descending diagonal

$$X_N = T_n(f) = \begin{bmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{1-n} \\ A_1 & A_0 & A_{-1} & & \vdots \\ A_2 & A_1 & A_0 & \ddots & A_{-2} \\ \vdots & & \ddots & \ddots & A_{-1} \\ A_{n-1} & & A_2 & A_1 & A_0 \end{bmatrix}$$

where

$$A_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \quad k \in \mathbb{Z}.$$

The function $f(\theta)$ is known as the symbol of T_n

Toeplitz, cont.

- In the FE setting, a d-dimensional problem with constant coefficients, discretized on quadrilateral FEs of degree p is a d level block valued Toeplitz with blocks of size, $s = p^d$.
- Each block $A_k, k = 1 \cdots n$ could be Toeplitz in itself. In this case T_n is multilevel Toeplitz.
- For f real-valued continuous, the eigenvalues of $T_n(f)$ for large n are approximate evaluations of f, over a grid of the form $\{x_j^{(n)}\}$, $x_j^{(n)} = -\pi + \frac{2\pi j}{n}, \quad j = 1, \cdots, n$.
- If the symbol f is an s-by-s matrix, the eigenvalues of T_n are described by the eigenvalues of n/s equispaced evaluations of f. If T_n is non-Hermitian, the same holds with the singular values of T_n.

Toeplitz, cont.

In the PDE setting, the system matrix X_N is Toeplitz
$X_N = T_n(f) + E_N$

- E_N is a sequence with symbol equal to zero and this kind of sequences occurs as well when treating variable coefficient PDEs.
- X_N has the same symbol as T_n.
- The generated uniform sampling of the symbol of T_n is different by a small margin from the eigenvalues of X_N.
- There may be very few outliers which are not covered by the symbol of T_n.

Generalized locally Toeplitz (GLT)

- The approximation of PDEs with non-constant coefficients, general domains, non-uniform gridding by local methods (FDM, FEM, IgA, etc), under very mild assumptions leads to a GLT sequence.

- Algebraic manipulation of GLT sequences result in a GLT sequence with the symbol constructed using the same algebraic manipulation on the symbols of the GLT sequences.

- If a GLT sequence is the result of a tensor product of two GLT sequences, e.g. $T_n(f) = T_{n_1}(\widetilde{f}) \otimes T_{n_2}(\widehat{f})$ then the symbol f is a two level symbol $f(\theta_1, \theta_2) = \widetilde{f}(\theta_1) \otimes \widehat{f}(\theta_2)$.

Example

$$\begin{cases} -(\kappa(x)u')' + v' & = & g_1(x) \\ u' - \rho v & = & g_2(x) \end{cases}, \quad \mathcal{A} = \begin{bmatrix} K & B^T \\ B & -\rho M \end{bmatrix},$$

$$K = \begin{bmatrix} \kappa_0 + \kappa_1 & -\kappa_1 & & & \\ -\kappa_1 & \kappa_1 + \kappa_2 & -\kappa_2 & & \\ & \kappa_2 & \ddots & \ddots & \\ & & \ddots & \ddots & -\kappa_n \\ & & & -\kappa_n & \kappa_n + \kappa_{n+1} \end{bmatrix} + E_N, \quad \kappa_i = \kappa(x_i) \ ,$$

$$f_K(\theta) = 2 - 1e^{\hat{\imath}\theta} - 1e^{-\hat{\imath}\theta} = 2 - 2\cos(\theta),$$

$$K = \kappa(x)\, T_n\big(f_K(\theta)\big).$$

Example

$$M = \frac{h^2}{6}\begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{bmatrix}, \quad B = h\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}.$$

$$f_M(\theta) = 4 + 1e^{\hat{\imath}\theta} + 1e^{-\hat{\imath}\theta} = 4 + 2\cos(\theta),$$

$$M = \frac{h^2}{3}T_n\big(f_M(\theta)\big),$$

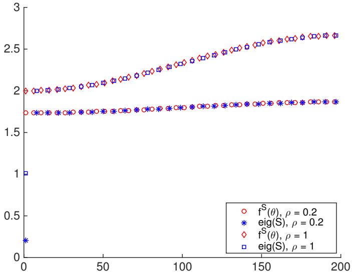
$$B^T = hT_n\big(1 - e^{-\hat{\imath}\theta}\big), \quad B = hT_n\big(1 - e^{\hat{\imath}\theta}\big).$$

Example

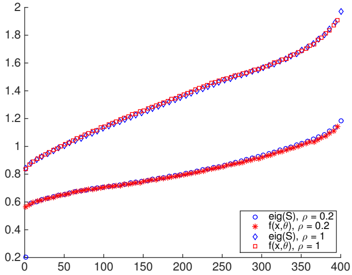
$$S = \rho M + B^T K^{-1} B.$$

$$\begin{aligned}\frac{1}{h^2} f^S(x,\theta) =& \frac{\rho}{3} T_n\big(2+\cos(\theta)\big) + \\ & T_n\big(1-e^{\hat{\imath}\theta}\big) \frac{1}{\kappa(x)\, T_n\big(2-2\cos(\theta)\big)} T_n\big(1-e^{-\hat{\imath}\theta}\big) \\ =& \frac{\rho}{3} T_n\big(2+\cos(\theta)\big) + \frac{1}{\kappa(x)}.\end{aligned}$$

Example



$\kappa(x) = \kappa_0$



$\kappa(x) = 1 + x$

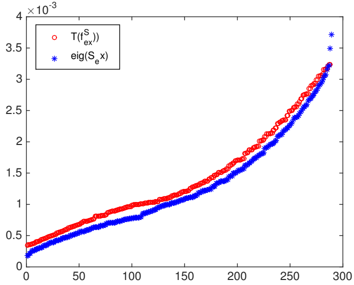
Special attention to B in $\mathcal{A}$

$$\mathcal{A} = \begin{bmatrix} K & B^T \\ B & -M \end{bmatrix}$$

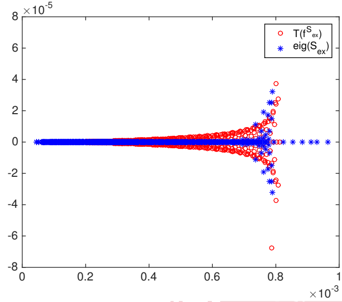
- For discretizations, Q1Q1, B is a rectangular matrix.
- We can view B as the result of a downsampling of some square matrix $\widetilde{B}$;
$$B = H^T \widetilde{B}.$$
- Use $\widetilde{B}$ in constructing the symbol of the exact Schur complement as
$$S_{\mathcal{A}} = \rho M + H^T \widetilde{B} K^{-1} \widetilde{B}^T H.$$
- Consider the effect of the multiplication by H and H^T on the symbol of $\widetilde{B} K^{-1} \widetilde{B}^T$.

Exact Schur complement - Elasticity

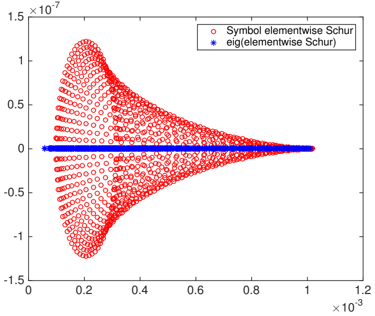
symmetric

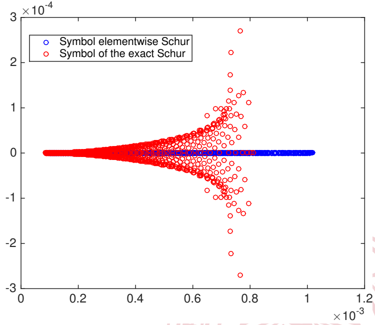


nonsymmetric



Additive Schur complement approximation; Q1Q1





Conclusion

Conclusion

- In the case of constant coefficients, the exact Schur complement of a saddle point matrix is analyzed using GLT sequences.
- The element-wise Schur complement approximation of the matrices is studied using GLT sequences in case of elasticity.
- The GLT tool has to be considered as an extension for matrix analysis.
- Future work:
  - Use carefully the symbol to design high quality preconditioner for Blocks K and $S_{\mathcal{A}}$
  - Navier-Stokes and other vector problems to be considered, with the idea of using the spectral information and the symbol to obtain faster and more robust (preconditioned) iterative solvers.

Thank you!