Adaptive-Shape *hp*-FEM for Multi-Scale Problems

Harri Hakula

Helsinki University of Technology (TKK), 02015 TKK, Finland,
Harri.Hakula@tkk.fi

Abstract. In the finite element method (FEM) the accuracy of the solution is controlled either by the mesh size or the polynomial degree of the elements or both. In *hp*-FEM it is often advantageous to employ geometric mesh grading together with varying polynomial order of the elements. Conforming strongly graded meshes are difficult to generate automatically. The fundamental idea behind the adaptive-shape FEM is to automatically adapt the basis functions to non-conforming meshes thus resulting in a fully continuous method. This allows us to use highly graded non-conforming meshes without discontinuous methods.

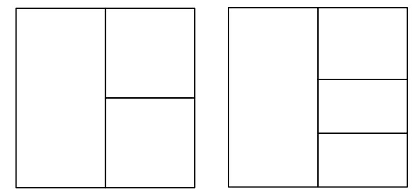
1 Introduction

Here the *hp*-FEM as defined by Szabo and Babuska in [1] is considered. The idea of modifying the method to accomodate the non-conforming meshes appears first in work by Demkowicz and others in [2] and is further expanded in their subsequent papers. The new approach outlined here does not require any changes to the standard *hp*-FEM infrastructure and can be added to any existing system with reasonable effort. Solin and others in [3] discuss the implementation issues of the standard *hp*-FEM including the 3D case. In this paper only the 2D adaptive shapes are discussed.

2 Adaptive Shapes

Let us consider the basic construction of the adaptive shapes with an example. In Fig. 1 two non-conforming meshes are shown. In the left-hand side case (Fig. 1a) the reference element is modified so that the shape functions associated with the split edge are replaced with those conforming to the shapes of the *neighbouring* elements. Detection of the hanging nodes and the proper handling of the split edges can be expensive and difficult to do robustly in practice.

For p-type quadrilaterals the number of the shape functions is $\frac{1}{2}(p^2+3p+6), p \geq 4$. In this example there will be $(p-1)+1 = p$ extra shapes corresponding to one new edge and node, respectively. So, at $p=4$ the split element will have five nodes and edges, and one bubble, in total $5+5\cdot(4-1)+1 = 21$ instead of the ususal 17 shapes. These shapes are shown in Fig. 2. Note that the nodal shape associated with the new node is shown in two parts (Fig. 2e and Fig. 2f).



(a) Mesh 1 (b) Mesh 2

Fig. 1: Two admissible meshes.

2.1 Integration

Let us denote the shape functions as ϕ_j. On the reference element $\hat{K}$ the supports of the shapes are not necessarily the same, that is, in general

$$\text{supp } \phi_i \neq \text{supp } \phi_j, \quad i \neq j.$$

Thus, the intersections of the supports must be computed for every inner product $\langle \cdot, \cdot \rangle$:

$$\int_{\hat{K}} \langle \phi_i, \phi_j \rangle \ da = \int_{\Omega(\phi_i,\phi_j)} \langle \phi_i, \phi_j \rangle \ da, \tag{1}$$

where $\Omega(\phi_i,\phi_j) = \text{supp } \phi_i \cap \text{supp } \phi_j$, the integration domain.

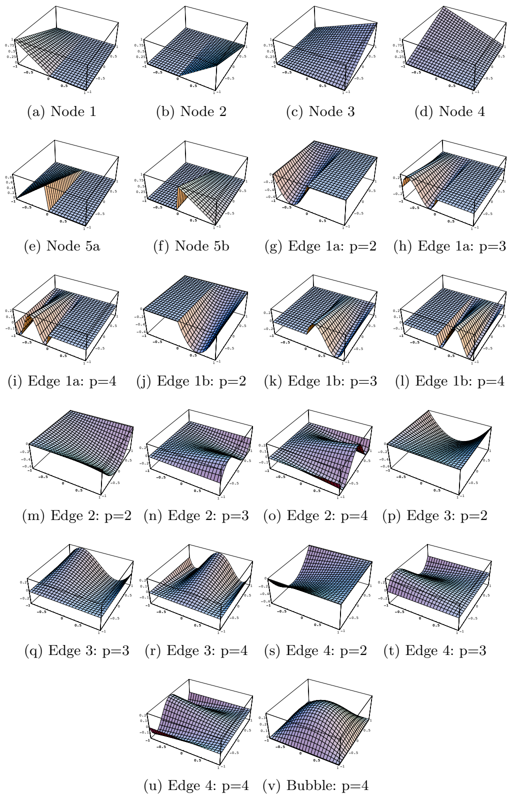
We choose the supports so that for every edge e of the reference element $\hat{K}$ the identity,

$$\bigcup_i \text{supp } \phi_i^e = \hat{K} \tag{2}$$

where ϕ_i^e are the shapes associated with the edge e, holds. On quadrilaterals, every support is simply a rectangle (See Fig. 2), and even in the case of multiple split edges, computation of $\Omega(\phi_i,\phi_j)$ is straightforward. On triangles, however, the situation is more complex.

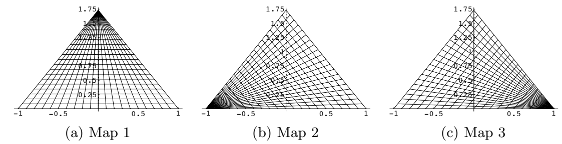
In Fig. 3 three images of the Cartesian grid over a quadrilateral reference element under bilinear maps that share the characteristic that one of the edges collapses onto a point in the triangle are shown. Using these maps, we can construct every conceivable configuration for the reference triangle as a map from the quadrilateral one.

In Fig. 4 two examples of split edge configurations on triangles are shown. In Fig. 4a the base has been split in two and the right-hand side in three parts. Finally, in Fig. 4b the integration domains for a 7×7 split configuration are illustrated.



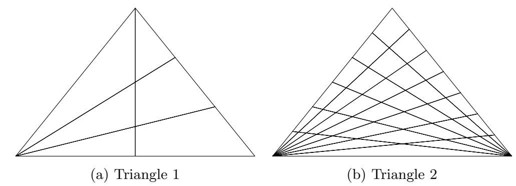
(a) Node 1 (b) Node 2 (c) Node 3 (d) Node 4
(e) Node 5a (f) Node 5b (g) Edge 1a: p=2 (h) Edge 1a: p=3
(i) Edge 1a: p=4 (j) Edge 1b: p=2 (k) Edge 1b: p=3 (l) Edge 1b: p=4
(m) Edge 2: p=2 (n) Edge 2: p=3 (o) Edge 2: p=4 (p) Edge 3: p=2
(q) Edge 3: p=3 (r) Edge 3: p=4 (s) Edge 4: p=2 (t) Edge 4: p=3
(u) Edge 4: p=4 (v) Bubble: p=4

Fig. 2: Quadrilateral with one split edge: p=4.



(a) Map 1 (b) Map 2 (c) Map 3

Fig. 3: Triangle maps.



(a) Triangle 1 (b) Triangle 2

Fig. 4: Triangle with two split edges.

3 Conclusions

There are many problems where exponential grading is advantageous. The strong layers in shell problems are of particular interest here. The adaptive-shape version of the *hp*-FEM presented here is a practical variant for such problems.

All modifications to the standard method are on the reference element. Therefore adaptive shapes are relatively simple to add to existing systems. The hidden cost is in the higher complexity of the necessary computational geometry.

Thorough mathematical analysis is still lacking. For non-curved meshes simple estimates can be derived from the standard theory.

References

1. Szabo, B., Babuska, I.: Finite Element Analysis. John Wiley & Sons, 1991.
2. Demkowicz, L. F., Oden, J. T., and Rachowicz, W., Toward a Universal h-p Adaptive Finite Element Strategy, Part 1. Constrained Approximation and Data Structure. CMAME **77** (1989) 79–112.
3. Solin, P., Segeth, K., and Dolezel, I.: Higher-Order Finite Element Methods. Chapman & Hall / CRC Press, 2003.