Recursive Blocked Algorithms for Solving Periodic Triangular Sylvester-type Matrix Equations

Robert Granat, Isak Jonsson, and Bo Kågström

Department of Computing Science and HPC2N, Umeå University,
SE-901 87 Umeå, Sweden,
{granat,isak,bokg}@cs.umu.se

Abstract. Recently, recursive blocked algorithms for solving one-sided and two-sided Sylvester-type equations were introduced [7–9]. This elegant yet simple technique enables an automatic variable blocking that has the potential of matching the memory hierarchies of today's HPC systems. The main parts of the computations are performed as level 3 general matrix multiply and add (GEMM) operations. In this presentation we extend our algorithms to also cover periodic Sylvester-type equations.

Keywords: Sylvester-type matrix equations, periodic matrix equations, recursion, level 3 BLAS, superscalar.

1 Introduction

In this presentation, we continue our work by developing algorithms for solving periodic Sylvester-type equations using recursion. One such example is the *periodic Sylvester equation*

$$\begin{aligned} A_k X_k - X_{k+1} B_k &= C_{k+1}, \qquad k = 1, \ldots, p-1 \\ A_p X_p - X_1 B_p &= C_1, \end{aligned}$$

where p is the periodicity of the matrix sequences, such that $A_{k+p} = A_k, B_{k+p} = B_k$ and $C_{k+p} = C_k$ [12, 14].

Here, we focus on recursive blocked algorithms for solving triangular periodic matrix equations, i.e., the matrix sequences A_k and B_k for $k = 1, \ldots, p$ are assumed to be in *periodic real Schur form* (PRSF) [2, 6]. This means that $p-1$ of the matrices in each sequence are upper triangular and one matrix in each sequence, say A_r for A_k and B_s for B_k, $1 \le r, s \le p$, is quasi-triangular. The products of conforming diagonal blocks of the matrix sequences A_k and B_k contain the eigenvalues of the matrix products $A_1 A_2 \cdots A_p$ and $B_1 B_2 \cdots B_p$, respectively, where the 1×1 and 2×2 blocks on the main block diagonal of A_r and B_s corresponds to the real and complex conjugate pairs of eigenvalues of these matrix products. Triangular matrix equations appear naturally in estimating the condition numbers of matrix equations and different eigenspace computations, including decoupling and stability analysis. To solve a triangular matrix equation is also a major step in the classical Bartels-Stewart method [1]. Periodic Sylvester-type matrix equations also appear in the context of eigenvalue reordering for computation and condition estimation of periodic invariant (deflating) subspaces of a matrix (pair) sequence [4, 5].

2 Recursive algorithms for periodic triangular matrix equations

Our research includes recursive blocked algorithms for solving the most common one-sided and two-sided triangular periodic Sylvester-type matrix equations. In this contribution, we focus on one-sided equations, namely the periodic continuous-time standard Sylvester and Lyapunov equations and a periodic generalized coupled Sylvester equation (see Table 1).

By using and reusing recursive templates, we can solve all matrix equations listed in Table 1 utilizing only a small set of subroutines. By this, we mean that, e.g., the PLYCT problem can be largely solved by the PSYCT routine. Therefore, the efforts of optimizing the implementation can be concentrated on a few core routines.

Name	Mnemonic	Matrix equation
Periodic standard Sylvester	PSYCT	$\begin{cases} A_k X_k - X_{k+1} B_k = C_{k+1} \\ A_p X_p - X_1 B_p = C_1 \end{cases}$
Periodic standard Lyapunov	PLYCT	$\begin{cases} A_k X_k + X_{k+1} A_k^T = C_{k+1} \\ A_p X_p + X_1 A_p^T = C_1 \end{cases}$
Periodic generalized coupled Sylvester	PGCSY	$\begin{cases} A_k X_k - Y_k B_k = C_{k+1} \\ D_k X_{k+1} - Y_k E_k = F_{k+1} \\ A_p X_p - Y_p B_p = C_1 \\ D_p X_1 - Y_p E_p = F_1 \end{cases}$

Table 1. Considered one-sided periodic Sylvester-type matrix equations. Here, p is the periodicity of the matrix sequences and $1 \leq k < p$.

The periodic matrix sequences are stored as 3-dimensional arrays, where the third dimension is the periodicity p of the matrix equation. The successive recursive splittings are performed on the 3-dimensional arrays explicitly, leading to new types of data locality issues.

Besides high-performance issues of the new algorithms, we also investigate accuracy issues including the solution of ill-conditioned periodic matrix equations and the solution of matrix equations with a large periodicity compared to the matrix dimensions. An important difference between non-periodic and periodic Sylvester-type matrix equations is the hard data dependency introduced from the periodicity of the problem. We discuss the impact of this data dependency and how to avoid degradation in speed from it. This is based on work done with the superscalar kernels that were developed for the RECSY library [7–9].

Different implementation issues are discussed, including when to end the recursion and the design of optimized and robust superscalar kernels for solving leaf-node triangular periodic matrix equations efficiently. Using the superscalar kernels, the new algorithms are able to solve periodic Sylvester-type equations with large periods but with small dimensions of the matrices (M, N). Uniprocessor and SMP parallel performance and accuracy results of our recursive blocked algorithms are presented.

Acknowledgements

This research was conducted using the resources of the High Performance Computer Center North (HPC2N). Financial support has been provided by the Swedish Research Council under grant VR 621-2001-3284 and by the Swedish Foundation for Strategic Research under grant A3 02:128.

References

1. R.H. Bartels and G.W. Stewart. Solution of the equation $AX + XB = C$. *Comm. Assoc. Comput. Mach.*, 15:820–826, 1972.
2. A.W. Bojanczyk, G. Golub and P. Van Dooren. The Periodic Schur Decomposition. Algorithms and Applications. In *Proceedings SPIE Conference* (F.T. Luk, Ed). Vol 1770, pp. 31–42, 1992.
3. E. Elmroth, F. Gustavson, I. Jonsson and B. Kågström. Recursive Blocked Algorithms and Hybrid Data Structures for Dense Matrix Library Software. *SIAM Review*, 26(1):3-45, 2004.
4. R. Granat and B. Kågström. Direct eigenvalue reordering in a product of matrices in extended periodic Schur form. Report UMINF-05.05, Department of Computing Science, Umeå University, Umeå, Sweden, 2005. To appear in *SIAM J. Matrix Anal. Appl.*
5. R. Granat, B. Kågström, D. Kressner. Reordering the Eigenvalues of a Periodic Matrix Pair with Applications in Control. Accepted for CACSD 2006.
6. J. J. Hench and A. J. Laub. Numerical solution of the discrete-time periodic Riccati equation. *IEEE Trans. Automat. Control*, 39(6):1197–1210, 1994.
7. I. Jonsson and B. Kågström. Recursive blocked algorithms for solving triangular systems — Part I: One-sided and coupled Sylvester-type matrix equations. *ACM Trans. Math. Softw.*, 28(4):392–415, December 2002.
8. I. Jonsson and B. Kågström. Recursive blocked algorithms for solving triangular systems — Part II: Two-sided and generalized Sylvester and Lyapunov matrix equations. *ACM Trans. Math. Softw.*, 28(4):416–435, December 2002.
9. I. Jonsson and B. Kågström. RECSY — A High Performance Library for Sylvester-Type Matrix Equations. In Kosch et al. (eds), *Euro-Par 2003, Parallel Processing*, Lecture Notes in Computer Science, Vol 2790, pp 810–819, Springer-Verlag, 2003.
10. B. Kågström and P. Poromaa. LAPACK–Style Algorithms and Software for Solving the Generalized Sylvester Equation and Estimating the Separation between Regular Matrix Pairs, *ACM Trans. Math. Software*, 22 (1996), pp. 78–103.

11. B. Kågström and L. Westin. Generalized Schur methods with condition estimators for solving the generalized Sylvester equation. *IEEE Trans. Automat. Control*, 34(4):745–751, 1989.
12. J. Sreedhar and P. Van Dooren, A Schur approach for solving some matrix equations, In *Systems and Networks: Mathematical Theory and Applications* (U. Helmke, R. Menniken and J. Saurer, Eds.). Vol. 77 of *Mathematical Research,* pp. 339–362, 1994.
13. A. Varga. Periodic Lyapunov equations: some applications and new algorithms. *Internat. J. Control*, 67(1):69–87, 1997.
14. A. Varga and P. Van Dooren. Computational methods for periodic systems. *Prepr. IFAC Workshop on Periodic Control Systems*, Como, Italy, pp. 177–182, 2001.