Edge stabilization and multiscale methods

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Abstract. We give a variational multiscale interpretation of a method recently introduced by the authors, the edge stabilization method using the jump in discrete gradient across element edges (faces in 3D) for the purpose of stabilization of Galerkin approximations of convection dominated flows. The ideas are exemplified using convection–diffusion as a model.

1 Introduction

Consider the solution of the convection–dominated convection–diffusion problem

$$\beta \cdot \nabla u - \varepsilon \nabla^2 u + \sigma u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

The usual variational formulation of this problem is to seek $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where $(\cdot,\cdot)$ denotes the scalar product in L_2 with norm $\|\cdot\|$, and

$$a(u, v) := (\varepsilon\, \nabla u, \nabla v) + b(u, v), \quad b(u, v) := (\beta \cdot \nabla u + \sigma u, v).$$

Under the usual assumption

$$\sigma - \frac{1}{2}\nabla \cdot \beta \geq c_0 > 0$$

$a(u, v)$ is coercive and (2) has a unique solution.

Standard Galerkin finite element methods consists in choosing a $V^h \subset H_0^1(\Omega)$ and to seek $u_h \in V^h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h. \tag{3}$$

This yields centered schemes from a finite difference perspective, which are well known to exhibit poor stability properties.

2 Gradient jump stabilization

In [1] a method based on the jump in gradients across element edges was proposed for the purpose of allleviating the convective instability of the Galerkin method. The idea is to add, to (3), a term of the type

$$\begin{aligned} J(u_h, v_h) &= \sum_K \frac{1}{2} \int_{\partial K} \gamma h_{\partial K}^2 [\nabla u_h] \cdot [\nabla v_h] \, ds \\ &= \sum_K \frac{1}{2} \int_{\partial K} \gamma h_{\partial K}^2 [n \cdot \nabla u_h] \, [n \cdot \nabla v_h] \, ds. \end{aligned} \tag{4}$$

Here, $h_{\partial K}$ is the size of ∂K, $[q]$ denotes the jump of q across ∂K for $\partial K \cap \partial \Omega = \emptyset$, $[q] = 0$ on $\partial K \cup \partial \Omega$, n is the outward pointing unit normal to K, and γ is a constant. It was shown in [1] that the term $J(u_h, v_h)$ gives control of the quantity $\|h_K^{1/2} \beta \cdot \nabla u_h\|^2$ which is the crucial point in the standard finite element stability proofs, as used, e.g., for the streamline diffusion methods or the discontinuous Galerkin methods applied to convective problems. An important step in the analysis of [1] was the proof that there exists some $\zeta \geq \zeta_0 > 0$ such that

$$\|h_K^{1/2} (\pi_h \beta \cdot \nabla u_h - \beta \cdot \nabla u_h)\|^2 \leq \zeta J(u_h, u_h), \tag{5}$$

where π_h is a suitable interpolant (the Clément interpolant in [1]).

3 Subgrid viscosity

Guermond [3] suggested the use of a fine scale viscosity operator acting only on the fine, unresolved, scales in a computation. He made use of a decomposition of a given $v_h \in V^h$ into

$$v_h = v_H + \tilde{v}_h, \quad v_H \in V^H, \ \tilde{v}_h \in \tilde{V}^h,$$

where $\tilde{V}^h$ represents a space with higher resolution than V^H. The decomposition corresponds to a decomposition of V^h into

$$V^h = V^H \oplus \tilde{V}^h.$$

The idea of [3] was then to apply an artificial viscosity only to the unresolved scales, viz.: Find $u_h \in V^h$ such that

$$(c_0 h_K \nabla \tilde{u}_h, \nabla \tilde{v}_h) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h, \tag{6}$$

or The decomposition can be defined by hierarchical meshes or, alternatively, by adding bubble functions, internal to the elements, on a given mesh, in which case $\tilde{v}_h$ represents the bubbles. If we additionally consider an orthogonal decomposition so that $(u_H - \tilde{u}_h, v_H) = 0$, $\forall v_H \in V^H$, or $u_H = P_H \tilde{u}_h$, the method can alternatively be written: find $u_h \in V^h$ such that

$$(c_1 \nabla (I - P_H) u_h, \nabla (I - P_H) v_h) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h, \tag{7}$$

yielding control of the term $\|\nabla (I - P_H) u_h\|$.

4 Edge stabilization and subgrid viscosity

The subgrid viscosity idea of [3] was subsequently reinterpreted by Layton [5] and John and Kaya [4], in both cases using a mixed method with auxiliary variables. In the case of convection–diffusion, Layton [5] introduced the auxiliary variable $p = \nabla u$. Now, if the discrete space Q^H for the approximation p_H of p has the property that $\nabla v_h \in Q^H, \forall v_h \in V^h$, then p_H can be directly eliminated in the discrete problem, but assume that Q^H is not big enough to ensure this. Layton then used the L_2–projection P_H onto the coarse scale and set $p_H = P_H \nabla u_h$ and modified (3) to seeking $u_h \in V^h$ such that

$$c_2\left((\nabla u_h, \nabla v_h) - (P_H \nabla u_h, \nabla v_h)\right) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h, \tag{8}$$

an idea remarkably close to that of Codina [2]. By the orthogonality of the L_2–projection, this can be written: find $u_h \in V^h$ such that

$$c_2((I - P_H)\nabla u_h, (I - P_H)\nabla v_h) + a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V^h, \tag{9}$$

giving control of the term $\|(I - P_H)\nabla u_h\|$. This is similar, but not equivalent, to (7). With specific choices of the discrete spaces, the correspondence can nevertheless become exact, cf. [5].

The same idea was used in [4] for the purpose of large-eddy turbulence modeling.

At this point, we can compare the subgrid viscosity models discussed above with the gradient jump stabilization method. In the latter case we would choose $V^H = V^h$ and thus make the subgrid diffusion act only on the scales that are not resolved on the space V^h. Applying (5) we obtain an interior penalty interpretation of the control of the term $\|(I - P_H)\nabla u_h\|$, with P_H replaced by π_h,

$$\|(I - \pi_h)\nabla u_h\|^2_\Omega \leq C \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K \, [\nabla u_h] \cdot [\nabla u_h] \, ds$$

and we conclude that a possible subgrid modeling term would be

$$j_T(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K \setminus \partial \Omega} c_3 h_K \, [\nabla u_h] \cdot [\nabla v_h] \, ds$$

where c_3 is at our disposal. Note that the choice $c_3 = \gamma h_K$ gives us a term which is asymptotically equivalent to the face penalty operator.

For sufficiently high polynomial degree there exists a C^1 subspace of V_h with approximation properties. It follows that the solution may be decomposed into one C^1 part which is untouched by the stabilizing terms and another C^0 part which is penalized. We conclude that the method enjoys the scale separation property characteristic for subgrid models as proposed in [3] by polynomial order rather than by hierarchic meshes.

References

1. Burman, E., Hansbo, P.: Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems, Comput. Methods Appl. Mech. Engrg. **193** (2004) 1437–1453.
2. Codina, R., Blasco, J,: Stabilized finite element method for the transient Navier-Stokes equations based on a pressure gradient projection, Comput. Methods Appl. Mech. Engrg. **182** (2000) 277–300.
3. Guermond, J.L.: Stabilization of Galerkin approximations of transport equations by subgrid modeling. M2AN Math. Model. Numer. Anal. **33** (1999) 1293–1316.
4. John, V., Kaya, S.: A finite element variational multiscale method for the Navier-Stokes equations. SIAM J. Sci. Comput. **26** (2005) 1485–1503.
5. Layton, W.: A connection between subgrid scale eddy viscosity and mixed methods, Appl. Math. Comp. **133** (2002) 147–157.